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LUDWIK BYSZEWSKI*, TERESA WINIARSKA*

CONTINUOUS DEPENDENCE OF MILD SOLUTIONS, ON INITIAL NONLOCAL DATA, OF THE NONLOCAL EVOLUTION CAUCHY PROBLEMS

CIĄGŁA ZALEŻNOŚĆ CAŁKOWYCH ROZWIĄZAŃ OD NIELOKALNYCH WARUNKÓW POCZĄTKOWYCH, NIELOKALNYCH EWOLUCYJNYCH ZAGADNIEŃ CAUCHY'EGO

Abstract

The aim of the paper is to prove two theorems on continuous dependence of mild solutions, on initial nonlocal data, of the nonlocal Cauchy problems. For this purpose, the method of semigroups and the theory of cosine family in Banach spaces are applied. The paper is based on publications [1–5].

Keywords: *evolution Cauchy problems*, *continuous dependence of solutions*, *nonlocal conditions*

Streszczenie

W artykule udowodniono dwa twierdzenia o ciągłej zależności rozwiązań całkowych od nielokalnych warunków początkowych, nielokalnych zagadnień Cauchy'ego. W tym celu zastosowano metodę półgrup i teorię rodziny cosinus w przestrzeniach Banacha. Artykuł bazuje na publikacjach [1–5].

Słowa kluczowe: *ewolucyjne zagadnienia Cauchy'ego*, *ciągła zależność rozwiązań*, *warunki nielokalne*

* Prof. Ludwik Byszewski, prof. Teresa Winiarska, Institute of Mathematics, Faculty of Physics, Mathematics and Computer Science, Cracow University of Technology.

Part I

Continuous dependence of mild solutions, on initial nonlocal data, of the nonlocal evolution Cauchy problem of the first order

1. Introduction to Part I

In this part of the paper, we assume that E is a Banach space with norm $\|\cdot\|$ and $-A$ is the infinitesimal generator of a $\mathcal{C}_0$ semigroup $\{T(t)\}_{t \geqslant 0}$ on E.

Throughout this part of the paper, we use the notation:

$$I = [0, a], \quad \text{where} \quad a > 0,$$

$$M = \sup\left\{\|T(t)\|,\ t \in I\right\}$$

and

$$X = \mathcal{C}(I, E).$$

Let p be a positive integer and $t_1, \ldots, t_p$ be given real numbers such that $0 < t_1 < \ldots < t_p$. Moreover, let C_i $(i = 1, \ldots, p)$ be given real numbers and

$$K := \sum_{i=1}^{p} |C_i|.$$

Consider the evolution nonlocal Cauchy problem of the first order

$$u'(t) + Au(t) = f(t), \quad t \in I \setminus \{0\}, \tag{1.1}$$

$$u(0) + \sum_{i=1}^{p} C_i u(t_i) = x_0, \tag{1.2}$$

where $f\colon I \to E$ and $x_0 \in E$.

In this part of the paper, we shall study a continuous dependence of a mild solution, on initial nonlocal data (1.2), of the nonlocal evolution Cauchy problem (1.1)–(1.2). The definition of this solution will be given in the next section.

This part of the paper is based on publications [1, 3–5]. Particularly, see Theorem 43.1 from [4].

2. Theorem about a mild solution of the nonlocal evolution Cauchy problem of the first order

A function $u \in X$ and satisfying the equation

$$u(t) = T(t)x_0 - T(t)\left(\sum_{i=1}^{p} C_i u(t_i)\right) + \int_0^t T(t-s) f(s)\,ds, \quad t \in I, \tag{2.1}$$

is said to be a mild solution of the nonlocal Cauchy problem (1.1)–(1.2).

Theorem 2.1. *Assume that*:

(i) $f : I \to E$ *is continuous,*

(ii) $MK < 1$,

(iii) $x_0 \in E$.

Then the nonlocal evolution Cauchy problem (1.1)–(1.2) *has a unique mild solution.*
Proof. See [1], Theorem 3.1 and page 32. □

3. Continuous dependence of a mild solution, on initial nonlocal data (1.2), of the nonlocal Cauchy problem (1.1)–(1.2)

Theorem 3.1. *Let all the assumptions of Theorem 2.1 be satisfied. Suppose that u is a mild solution (satisfying* (2.1)*) from Theorem 2.1. Moreover, let* $v \in X$, *satisfying the equation*

$$v(t) = T(t)y_0 - T(t)\left(\sum_{i=1}^{p} C_i v(t_i)\right) + \int_0^t T(t-s)f(s)ds, \quad t \in I, \tag{3.1}$$

be the mild solution to the nonlocal problem

$$v'(t) + Av(t) = f(t), \quad t \in I \setminus \{0\},$$

$$v(0) + \sum_{i=1}^{p} C_i v(t_i) = y_0,$$

where $y_0 \in I$.

Then for an arbitrary $\varepsilon > 0$ *there is* $\delta > 0$ *such that if*

$$\|x_0 - y_0\| < \delta \tag{3.2}$$

and

$$\|u(t_i) - v(t_i)\| < \delta \quad (i = 1, \ldots, p) \tag{3.3}$$

then

$$\|u - v\|_X < \varepsilon. \tag{3.4}$$

Proof. Let ε be a positive number and let

$$\delta := \min\left\{\frac{\varepsilon}{2M}, \frac{\varepsilon}{2MKp}\right\}. \tag{3.5}$$

Observe that, from (2.1) and (3.1)

$$u(t) - v(t) = T(t)(x_0 - y_0) - T(t)\left(\sum_{i=1}^{p} C_i(u(t_i) - v(t_i))\right), \quad t \in I.$$

Consequently, by (3.2), (3.3) and (3.5),

$$\|u(t)-v(t)\| \leqslant M\|x_0-y_0\|+MK\sum_{i=1}^{p}\|u(t_i)-v(t_i)\| < M\delta+MKp\delta \leqslant \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon \quad \text{for} \quad t\in I.$$

Therefore, (3.2) holds. It means that a mild solution of the nonlocal Cauchy problem (1.1)–(1.2) is continuously dependent on the initial nonlocal data (1.2).

The proof of Theorem 3.1 is complete. □

Part II

Continuous dependence of mild solutions, on nonlocal data, of the nonlocal evolution Cauchy problem of the second order

4. Introduction to Part II

In the second part of the paper, we consider the nonlocal evolution Cauchy problem of the second order

$$u''(t)=Au(t)+f(t), \quad t\in I\setminus\{0\}, \tag{4.1}$$

$$u(0)=x_0, \tag{4.2}$$

$$u'(0)+\sum_{i=1}^{p}C_i u(t_i)=x_1, \tag{4.3}$$

where A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators from the Banach space E (with norm $\|\cdot\|$) into itself, $u : I \to E$, $f : I \to E$, $I = [0, a]$, $a > 0$, $x_0, x_1 \in E$, $C_i \in \mathbb{R}$ $(i = 1, \ldots, p)$ and $t_1, \ldots, t_p$ are as in Part I.

We will use the set

$$\tilde{E} := \{x \in E : C(t)x \text{ is of class } C^1 \text{ with respect to } t\}$$

and the sine family $\{S(t) : t \in \mathbb{R}\}$ defined by the formula

$$S(t)x := \int_0^t C(s)x\,ds, \quad x\in E,\ t\in\mathbb{R}.$$

In this part of the paper, we shall study a continuous dependence of a mild solution, on initial nonlocal data (4.2)–(4.3), of the nonlocal evolution Cauchy problem (4.1)–(4.3). The definition of this solution will be given in the next section.

The second part of the paper is based on publications [2, 4].

5. Theorem about a mild solution of the nonlocal Cauchy problem of the second order

A function $u \in \mathcal{C}^1(I, E)$ and satisfying the equation

$$u(t) = C(t)x_0 + S(t)x_1 - S(t)\left(\sum_{i=1}^{p} C_i u(t_i)\right) + \int_0^t S(t-s)f(s)ds, \quad t \in I, \tag{5.1}$$

is said to be a mild solution of the nonlocal Cauchy problem (4.1)–(4.3).

Theorem 5.1. *Assume that*:

(i) $f : I \to E$ *is continuous*,

(ii) $2CK < 1$, *where* $C := \sup\{\|C(t)\| + \|S(t)\| + \|S'(t)\| : t \in I\}$ *and* $K := \sum_{i=1}^{p} |C_i|$,

(iii) $x_0 \in \tilde{E}$ and $x_1 \in E$.

Then the nonlocal evolution Cauchy problem (4.1)–(4.3) *has a unique mild solution.*

Proof. See [2], Theorem 2.1. □

6. Continuous dependence of a mild solution on initial nonlocal data (4.2)–(4.3), of the nonlocal evolution Cauchy problem (4.1)–(4.3)

Theorem 6.1. *Let all the assumptions of Theorem 5.1 be satisfied. Suppose that u is a mild solution (satisfying* (5.1)*) from Theorem 5.1. Moreover, let v satisfying the equation*

$$v(t) = C(t)y_0 + S(t)y_1 - S(t)\left(\sum_{i=1}^{p} C_i v(t_i)\right) + \int_0^t S(t-s)f(s)ds, \quad t \in I, \tag{6.1}$$

be the mild solution of the nonlocal problem

$$v''(t) = Av(t) + f(t), \quad t \in I \setminus \{0\},$$

$$v(0) = y_0,$$

$$v'(0) + \sum_{i=1}^{p} C_i v(t_i) = y_1,$$

where $y_0 \in \tilde{E}$ *and* $y_1 \in E$.

Then for any arbitrary $\varepsilon > 0$ *there is* $\delta > 0$ *such that if*

$$\|x_0 - y_0\| < \delta, \quad \|x_1 - y_1\| < \delta \tag{6.2}$$

and

$$\|u(t_i) - v(t_i)\| < \delta \quad (i = 1, \ldots, p) \tag{6.3}$$

then

$$\|u - v\|_X < \varepsilon, \tag{6.4}$$

where $X = \mathcal{C}(X, E)$.

Proof. Let ε be a positive number and let

$$\delta := \min\left\{\frac{\varepsilon}{3C}, \frac{\varepsilon}{3CKp}\right\}. \tag{6.5}$$

Observe that, from (5.1) and (6.1),

$$u(t) - v(t) = C(t)(x_0 - y_0) + S(t)(x_1 - y_1) - S(t)\left(\sum_{i=1}^{p} C_i(u(t_i) - v(t_i))\right), \quad t \in I.$$

Consequently, by (6.2), (6.3) and (6.5),

$$\|u(t) - v(t)\| \leqslant C\|x_0 - y_0\| + C\|x_1 - y_1\| + CK\sum_{i=1}^{p}\|u(t_i) - v(t_i)\| <$$
$$< C\delta + C\delta + CKp\delta \leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \text{for} \quad t \in I.$$

Therefore, (6.4) holds. It means that a mild solution of the nonlocal Cauchy problem (4.1)–(4.3) is continuously dependent on the initial nonlocal data (4.2)–(4.3).

The proof of Theorem 6.1 is complete. □

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