TECHNICAL TRANSACTIONS | CZASOPISMO TECHNICZNE
FUNDAMENTAL SCIENCES | NAUKI PODSTAWOWE
2-NP/2014

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CYCLES CONTAINING SPECIFIED EDGES IN A GRAPH

CYKLE ZAWIERAJĄCE WYBRANE KRAWĘDZIE GRAFU

Abstract

The aim of this paper is to prove that if $s \geqslant 1$ and G is a graph of order $n \geqslant 4s+6$ satisfying

$$\sigma_2 \geqslant \frac{4n-4s-3}{3},$$

then every matching of G lies on a cycle of length at least $n-s$ and hence, in a path of length at least $n-s+1$.

Keywords: cycle, graph, hamiltonian cycle, hamiltonian path, matching, path

Streszczenie

W pracy udowodniono, że dla $s \geqslant 1$ w dowolnym grafie G rzędu $n \geqslant 4s+6$ spełniającym

$$\sigma_2 \geqslant \frac{4n-4s-3}{3},$$

każde skojarzenie jest zawarte w cyklu długości co najmniej $n-s$ i stąd w ścieżce długości co najmniej $n-s+1$.

Słowa kluczowe: cykl, cykl hamiltonowski, graf, skojarzenie, ścieżka, ścieżka hamiltonowska

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1. Introduction

We consider only finite graphs without loops and multiple edges. By V or $\mathrm{V}(G)$ we denote the vertex set of graph G and respectively, by E or $\mathrm{E}(G)$, the edge set of G. By $\mathrm{d}_G(x)$ or $\mathrm{d}(x)$ we denote *the degree of a vertex x in the graph G*.

In the proof we will only use oriented cycles and paths. Let C be a cycle and $x \in V(C)$, then x^- is *the predecessor of* x and x^+ is its *successor.*

Let us introduce the σ_k.

Definition 1.1. *Let G be a graph and $k \geqslant 0$.*

$$\sigma_k = \min\{\sum_{i=1}^{k} \mathrm{d}(x_i) : \{x_1, \dots, x_k\} \subset \mathrm{V}(G) \text{ and independent } \}$$

In 1960, O. Ore [5] proved the following:

Theorem 1.1. *Let G be a graph on $n \geqslant 3$ vertices. If G satisfies*

$$\sigma_2 \geqslant n$$

then G is hamiltonian.

The condition for degree sum in Theorem 1.1 is called *an Ore condition* or *a Ore type condition for graph G.*

The Ore condition for a graph G :

$$\sigma_2 \geqslant l$$

can also be written as:

$$\text{If } x, y \in \mathrm{V}(G),\ xy \notin \mathrm{E}(G), \quad \text{then:} \quad \mathrm{d}(x) + \mathrm{d}(y) \geqslant l.$$

There is also a similar condition, proved by V. Chvátal in [3], under which graph G has a hamiltonian path.

Theorem 1.2. *If G is a graph on $n \geqslant 3$ vertices satisfying*

$$\sigma_2 \geqslant n - 1, \tag{1.1}$$

then G has a hamiltonian path.

We shall call a set of k independent edges of graph G a *k-matching* or simply a *matching.*

About graphs with every k-matching in a hamiltonian cycle or path Las Vergnas obtained the following two results:

Theorem 1.3. *Let G be a graph on $n \geqslant 3$ vertices and let k be an integer such that $0 \leqslant k \leqslant \frac{n}{2}$. If G satisfies*

$$\sigma_2 \geqslant n + k - 1\,,$$

then every k-matching of G lies in a hamiltonian path.

Theorem 1.4. *Let G be a graph on $n \geqslant 3$ vertices and let k be an integer such that $0 \leqslant k \leqslant \frac{n}{2}$. If G satisfies*

$$\sigma_2 \geqslant n + k\,,$$

then every k-matching of G lies in a hamiltonian cycle.

K.A. Berman proved in [1] the following result conjectured by R. Häggkvist in [4].

Theorem 1.5. *Let G be graph of order n. If G satisfies*

$$\sigma_2 \geqslant n + 1,$$

then every matching lies in a cycle.

Now we shall define a family of graphs $\mathcal{G}_n$. If $\frac{n+2}{3}$ is an integer, $\mathcal{G}_n$ is a family of graphs:

$$G = \frac{n+2}{3}\,K_1 \star H,$$

where $\star$ denotes join and H is a graph of order $\frac{2n-2}{3}$ containing a perfect matching. Otherwise, $\mathcal{G}_n$ is empty.

In 1983 Wojda [6] proved the following Ore type theorem:

Theorem 1.6. *Let G be a graph on $n \geqslant 3$ vertices.If G satisfies*

$$\sigma_2 \geqslant \frac{4n-4}{3}\,.$$

Then every matching of G lies in a hamiltonian cycle or $G \in \mathcal{G}_n$.

In this paper, we shall find an Ore type condition under which every matching in a graph G lies in a cycle of length at least $n - s$ and hence, in a path of length at least $n - s + 1$.

For the notation and terminology not defined above, a good reference should be [2].

2. Results

We proved the following improvement of Theorem 1.6 for matchings.

Theorem 2.1. *Let $s \geqslant 1$ and let G be a graph of order $n \geqslant 4s+6$ satisfying*

$$\sigma_2 \geqslant \frac{4n-4s-3}{3}. \tag{2.1}$$

Then, every matching of G lies on a cycle of length at least $n-s$ and hence in a path of length at least $n-s+1$.

The special case of this theorem for $s=1$ is:

Corollary 2.2. *Let G be a graph of order $n \geqslant 10$, satisfying*

$$\sigma_2 \geqslant \frac{4n-7}{3}.$$

Then, every matching of G lies on a cycle of length at least $n-1$ and hence in a hamiltonian path.

Obviously for $k \geqslant \frac{n-2}{3}$, the bound for σ_2 is lower in Corollary 2.2 than the bound from Theorem 1.3.

Suppose that $s \geqslant 1$ is such that $n \geqslant 4s+6$ and $\frac{n+2s}{3} \geqslant 2$ is an integer.

Now consider the graph $G' = (\frac{n+2s}{3} - 1)\, K_1 * K_{\frac{2n-2s}{3}}$, where $*$ denotes the join of graphs.

We shall define a graph G'' as a graph obtained from G' by adding an external vertex x adjacent only to $\frac{2n-2s}{3} - 1$ vertices from $K_{\frac{2n-2s}{3}}$ i.e. we take $\mathrm{V}(G'') = \mathrm{V}(G') \cup \{x\}$, next we choose an arbitrary vertex $h_0 \in \mathrm{V}(K_{\frac{2n-2s}{3}})$ and we put $\mathrm{E}(G'') = \mathrm{E}(G') \cup \{xh : h \in \mathrm{V}(K_{\frac{2n-2s}{3}}) \setminus \{h_0\}\}$. Note theat G'' is a graph of order n.

Let $u \in \mathrm{V}(K_1)$, then $\mathrm{d}_{G''}(u) = \frac{2n-2s}{3} - 1$ and $\mathrm{d}_{G''}(x) + \mathrm{d}_{G''}(h_0) = \frac{4n-4s-3}{3}$, the graph G'' satisfies the assumptions of Theorem 2.1, but violates those of Theorem 1.6. So, Theorem 1.6 and Theorem 2.1 are independent.

It is easy to check that even Corollary 2.2 cannot be obtained as a corollary of Theorem 1.6 by adding to the graph G an external vertex x adjacent to all vertices and removing an edge from the hamiltonian cycle in $G \cup \{x\}$. In this case, $G \cup \{x\}$ does not satisfy the assumptions of Theorem 1.6.

Obviously, for $k \geqslant \frac{n-2}{3}$, the bound for σ_2 is lower in Theorem 2.2 than the bound from Theorem 1.3.

3. Proof

Proof of Theorem 2.1:

Take any matching M of G. Without loss of generality we can assume that M is maximal, i.e. for any matching M' of G if $M \subset M'$, then $M = M'$.

Observe that since $n \geqslant 4s + 6$, we have $\frac{4n-4s-3}{3} \geqslant n + 1$. If the graph G satisfies the assumptions of Theorem 2.1 then it also satisfies the assumptions of Theorem 1.5. From Theorem 1.5, we know that there is a cycle containing M.

Consider a cycle C containing M of maximal length. If $|\mathrm{V}(C)| \geqslant n - s$ the proof is finished. We suppose now that $|\mathrm{V}(C)| \leqslant n - s - 1$ and we give an arbitrary orientation to C.

Since M is maximal, the set $\mathrm{V}(G \setminus C)$ is independent.

Since $s \geqslant 1$ we have $|\mathrm{V}(G \setminus C)| \geqslant 2$ and therefore we have two vertices x and $y \in \mathrm{V}(G \setminus C)$ such that $xy \notin \mathrm{E}(G)$ and from (2.1) we have:

$$\mathrm{d}(x) + \mathrm{d}(y) \geqslant \frac{4n - 4s - 3}{3}. \tag{3.1}$$

Note that since $\mathrm{V}(G \setminus C)$ is independent we have:

$$\mathrm{d}_{G\setminus C}(x) = \mathrm{d}_{G\setminus C}(y) = 0. \tag{3.2}$$

On cycle C, consider a family of paths Q_i, $i \in \{1, \ldots, k\}$, obtained from C by the removal of the edges of matching M.

Note that:

$$\sum_{i=1}^{k} |\mathrm{V}(Q_i)| = |\mathrm{V}(C)|. \tag{3.3}$$

Since M is maximal, $|\mathrm{V}(Q_i)| = 2$ or $|\mathrm{V}(Q_i)| = 3$. (3.4)

Remark 3.1. *If $w \in \mathrm{V}(G \setminus C)$, then w cannot be adjacent to two consecutive vertices on any path Q_i, for $i \in \{1, \ldots, k\}$.*

Suppose that $w \in \mathrm{V}(G \setminus C)$ and we have a vertex $u \in \mathrm{V}(Q_i)$ such that $u+ \in \mathrm{V}(Q_i)$ and wu, $wu+ \in \mathrm{E}$. In this case, the cycle:

$$C' : \quad uwu^{+} \ldots u^{-}u$$

contains M and is longer than C, contradiction with the choice of C.

Case 1: $|\mathrm{V}(Q_i)| = 2$

From Remark 3.1, we know that $\mathrm{d}_{Q_i}(x) \leqslant 1$ and $\mathrm{d}_{Q_i}(y) \leqslant 1$ and so:

$$\mathrm{d}_{Q_i}(x) + \mathrm{d}_{Q_i}(y) \leqslant 2 = |\mathrm{V}(Q_i)|. \tag{3.5}$$

Case 2: $|\mathrm{V}(Q_i)| = 3$

In this case, from Remark 3.1 we know that $\mathrm{d}_{Q_i}(x) \leqslant 2$ and $\mathrm{d}_{Q_i}(y) \leqslant 2$. Note that if $Q_i : q_1^i q_2^i q_3^i$ it is possible that x and y are adjacent at the same time to q_1^i and q_2^i. From the above we have:

$$\mathrm{d}_{Q_i}(x) + \mathrm{d}_{Q_i}(y) \leqslant 4 = |\mathrm{V}(Q_i)| + 1. \tag{3.6}$$

Consider now the set $I = \{Q_i : \ |\mathrm{V}(Q_i)| = 3 \text{ and } Q_i : \ q_1^i q_2^i q_3^i\}$, $l = |I|$. Observe that since M is maximal we have:

1. The set $V_I = \{q_2^i : Q_i \in I\}$ is independent and $|V_I| = l$.
2. If $\mathrm{d}_{Q_i}(x) = 2$, then $xq_2^i \notin E$ and if $\mathrm{d}_{Q_i}(y) = 2$, then $yq_2^i \notin E$.

We have $l + 2$ independent vertices in $G : V_I \cup \{x, y\}$.

Thus:

$$\begin{aligned}\mathrm{d}_C(x) + \mathrm{d}_C(y) &= \sum_{i=1}^{k} \mathrm{d}_{Q_i}(x) + \mathrm{d}_{Q_i}(y) \\ &\leqslant \sum_{i=1}^{k} |\mathrm{V}(Q_i)| + l = |\mathrm{V}(C)| + l.\end{aligned} \tag{3.7}$$

From (3.2) and (3.7), we have:

$$\mathrm{d}(x) + \mathrm{d}(y) = \sum_{i=1}^{k} \mathrm{d}_{Q_i}(x) + \mathrm{d}_{Q_i}(y) \leqslant |\mathrm{V}(C)| + l. \tag{3.8}$$

Since $|\mathrm{V}(C)| \leqslant n - s - 1$ and M is maximal, we have $l \leqslant \frac{n-s-1}{3}$ and so:

$$\mathrm{d}(x) + \mathrm{d}(y) \leqslant |\mathrm{V}(C)| + l \leqslant n - s - 1 + \frac{n - s - 1}{3} = \frac{4n - 4s - 4}{3}. \tag{3.9}$$

Since $\mathrm{d}(x) + \mathrm{d}(y)$ is an integer, from (3.9), we have:

$$\mathrm{d}(x) + \mathrm{d}(y) \leqslant \left\lfloor \frac{4n - 4s - 4}{3} \right\rfloor < \frac{4n - 4s - 3}{3}, \tag{3.10}$$

a contradiction with (3.1) and the proof is finished.

$\square$

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