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THE EXISTENCE OF A WEAK SOLUTION OF THE SEMILINEAR FIRST-ORDER DIFFERENTIAL EQUATION IN A BANACH SPACE

ISTNIENIE SŁABEGO ROZWIĄZANIA SEMILINIOWEGO RÓWNANIA RÓŻNICZKOWEGO PIERWSZEGO RZĘDU W PRZESTRZENI BANACHA

A b s t r a c t

This paper is devoted to the investigation of the existence and uniqueness of a suitably defined weak solution of the abstract semilinear value problem $\dot{u}(t) = Au(t) + f(t, u(t))$, $u(0) = x$ with $x \in X$, where X is a Banach space. We are concerned with two types of solutions: weak and mild. Under the assumption that A is the generator of a strongly continuous semigroup of linear, bounded operators, we also establish sufficient conditions such that if u is a weak (mild) solution of the initial value problem, then u is a mild (weak) solution of that problem.

Keywords: operator, semigroup, weak solution

S t r e s z c z e n i e

Celem pracy jest przedstawienie twierdzenia o jednoznaczności i istnieniu słabego rozwiązania abstrakcyjnego semiliniowego równania różniczkowego $\dot{u}(t) = Au(t) + f(t, u(t))$, $u(0) = x$, gdzie $x \in X$, w przestrzeni Banacha X. W pracy rozważane są dwa typy rozwiązań: *weak* oraz *mild.* Przy założeniu, że operator A jest generatorem silnie ciągłej półgrupy operatorów liniowych i ograniczonych, podane zostały również warunki wystarczające na to aby rozwiązanie *weak (mild)* było rozwiązaniem *mild (weak)* tego zagadnienia.

Słowa kluczowe: operator, półgrupa, słabe rozwiązanie

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1. Introduction

For a real or complex Banach space X, X^* will denote its dual space. Let $\langle \cdot, \cdot \rangle$ be the duality pairing between X and its dual space X^*. For an operator A, $D(A)$ and A^* will denote its domain and the adjoint, respectively. We consider the abstract first-order initial value problem

$$\frac{d}{dt}u(t) = Au(t) + f(t, u(t)) \quad \text{for} \quad t \in (0, T], \tag{1.1}$$

$$u(0) = x, \tag{1.2}$$

where A is a densely defined, closed linear operator on the Banach space X, $x \in X$ and $f : [0, T] \times X \to X$.

Definition 1. *A function $u \in C([0, T]; X)$ is a weak solution of (1) on $[0, T]$ if for every $v \in D(A^*)$, the function $[0, T] \ni t \to \langle u(t), v \rangle$ is absolutely continuous on $[0, T]$ and*

$$\frac{d}{dt}\langle u(t), v \rangle = \langle u(t), A^* v \rangle + \langle f(t, u(t)), v \rangle \text{ a.e. on } [0, T]. \tag{1.3}$$

2. Preliminaries

Let A be a densely defined linear operator on a real or complex Banach space X, let $T > 0$ and let $g \in L^1(0, T; X)$. It is well known (see [1]) that

Theorem 2. *If A is the generator of a strongly continuous semigroup of bounded linear operators $\{S(t)\}_{t \geq 0}$ on X, and if $x \in X$, then the first order linear equation*

$$\dot{w}(t) = Aw(t) + g(t), \;\; t \in (0, T], \tag{2.1}$$

has a unique weak solution (see Definition 3) satisfying $w(0) = x$, and in this case, w is given by

$$w(t) = S(t)x + \int_0^t S(t - s)g(s)ds, \quad t \in [0, T]. \tag{2.2}$$

Definition 3. *A function $w \in C([0, T]; \; X)$ is a weak solution of (2.1) on $[0, T]$ if for every $v \in D(A^*)$, the function $\langle w(t), v \rangle$ is absolutely continuous on $[0, T]$ and*

$$\frac{d}{dt}\langle w(t), v \rangle = \langle w(t), A^* v \rangle + \langle g(t), v \rangle \text{ a.e. on } [0, T]. \tag{2.3}$$

When $x \in X$ is arbitrary, then unless $\{S(t)\}_{t \geq 0}$ and f have special properties, w given by (2.2) will not, in general, belong to $D(A)$ for $t \in (0, T]$, so that (2.1) does not even make sense.

3. Existence and uniqueness of a weak solution of the problem (1)—(2)

We start with the following

Theorem 4. *Let A be the infinitesimal generator of a C_0 semigroup $\{S(t)\}_{t\geq 0}$ of bounded linear operators on X, $u \in C([0,T];\ X)$ and $f(\cdot, u(\cdot)) \in L^1(0,T;\ X)$. If u is a weak solution of the equation* (1) *and $u(0) = x$, then u is a solution of the integral equation*

$$u(t) = S(t)x + \int_0^t S(t-s)f(s,u(s))ds, \quad t \in [0,T]. \tag{3.1}$$

A continuous solution u of the integral equation (3.1) will be called a mild solution if the initial value problem (1)–(2).

Proof. Let u be a weak solution of (1) satisfying $u(0) = x$. This implies that for any $v \in D(A^*)$

$$\frac{d}{dt}\langle u(t), v\rangle = \langle u(t), A^*v\rangle + \langle f(t,u(t)), v\rangle \ a.e. \text{ on } [0,T]. \tag{3.2}$$

Let us put $g(t) := f(t,u(t))$ and $w(t) := S(t)x + \int_0^t S(t-s)g(s)ds$ for $t \in [0,T]$. Cleary, by Theorem 2, w is a unique weak solution of the problem

$$\begin{cases} \dot{w}(t) = Aw(t) + g(t), \ \ t \in (0,T], \\ w(0) = x. \end{cases} \tag{3.3}$$

By Definition 3,

$$\frac{d}{dt}\langle w(t), v\rangle = \langle w(t), A^*v\rangle + \langle g(t), v\rangle \ a.e. \text{ on } [0,T]. \tag{3.4}$$

Hence, by (3.2), the function u satisfies (3.4). By the uniqueness of the weak solution of the initial value problem (3.3)

$$u = w,$$

so

$$u(t) = w(t) = S(t)x + \int_0^t S(t-s)g(s)ds = S(t)x + \int_0^t S(t-s)f(s,u(s))ds$$

The proof of Theorem 4 is complete. □

The integral equation (3.1) does not necessarily admit a solution of any kind. However, if it has a continuous solution, then that function is a weak solution of the problem (1)–(2).

Theorem 5. *Let A be the infinitesimal generator of a C_0 semigroup $\{S(t)\}_{t\geq 0}$ of bounded linear operators on X, $u \in C([0,T];\ X)$ and $f(\cdot, u(\cdot)) \in L^1(0,T;\ X)$. If u is a solution of the integral equation (3.1), then u is a weak solution of the equation (1).*

Proof. By Theorem 2 the initial value problem

$$\begin{cases} \dot{w}(t) = Aw(t) + f(t, u(t)), & t \in (0,T], \\ w(0) = x \end{cases} \tag{3.5}$$

has exactly one weak solution given by $w(t) := S(t)x + \int_0^t S(t-s)f(s,u(s))ds$ for $t \in [0,T]$. By the assumption

$$u(t) = S(t)x + \int_0^t S(t-s)f(s,u(s))ds$$

for $t \in [0,T]$, so $w = u$ and u is the weak solution of (3.5). This completes the proof. $\square$

The main result of this paper is the following theorem

Theorem 6. *Let $f : [0,T] \times X \to X$ be continuous in t on $[0,T]$ and uniformly Lipschitz continuous on X. If A is the infinitesimal generator of a C_0 semigroup $\{S(t)\}_{t\geq 0}$ of bounded linear operators on X, then there exists for each $x \in X$ a unique weak solution u of (1) satisfying $u(0) = x$.*

Proof. By Theorem 6.1.2 [4] (page 184) (see [2], p. 77, [3], p. 87) the integral equation (3.1) has a unique solution $u \in C([0,T];\ X)$. From this, by Theorem 5, u is a weak solution of the equation (1) and $u(0) = x$. The uniqueness of u is a consequence of Theorem 4.

The proof is complete. $\square$

References

[1] Ball J.M., *Strongly continuous semigroups, weak solutions, and the variation of constant formula*, Proc. Amer. Math. Soc., 1977, 370–373.

[2] Hundertmark D., Meyries M., Machinek L., Schnaubelt R., *Operator Semigroups and Dispersive Equations*, 16th Internet Seminar on Evolution Equations, 2013.

[3] Goldstein J., *Semigroups of Linear Operators and Applications*, Oxford U. Press, New York 1985.

[4] Pazy A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer–Verlag, 1983.