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STRONG MAXIMUM PRINCIPLES FOR INFINITE IMPLICIT SYSTEMS OF PARABOLIC FUNCTIONAL-DIFFERENTIAL INEQUALITIES TOGETHER WITH NONLOCAL INEQUALITIES WITH INTEGRALS

MOCNE ZASADY MAKSIMUM DLA NIESKOŃCZONYCH UWIKŁANYCH UKŁADÓW PARABOLICZNYCH FUNKCJONALNO-RÓŻNICZKOWYCH NIERÓWNOŚCI Z NIELOKALNYMI NIERÓWNOŚCIAMI Z CAŁKAMI

A b s t r a c t

In this paper we consider infinite implicit systems of parabolic functional-differential inequalities together with nonlocal initial inequalities with integrals. For that systems we give strong maximum principles in relatively arbitrary (n + 1) – dimensional time space sets more general than the cylindrical domain.

Keyword: *infinite parabolic systems*, *implicit systems*, *nonlocal inequalities*, *strong maximum principles*

S t r e s z c z e n i e

W pracy rozważamy nieskończone uwikłane układy parabolicznych funkcjonalno-różniczkowych nierówności z nielokalnymi początkowymi nierównościami z całkami. Dla układów tych dowodzimy mocnych zasad maksimum we względnie dowolnych (n + 1) – wymiarowych zbiorach czasoprzestrzennych bardziej ogólnych niż obszar walcowy.

Słowa kluczowe: nieskończone układy paraboliczne, *układy uwikłane*, *nierówności nielokalne*, *mocne zasady maksimum*

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1. Introduction

In this paper we consider infinite implicit diagonal systems of non-linear parabolic functional-differential inequalities of the form

$$\begin{aligned} &F_i(x,t,u^i(x,t),u_t^i(x,t),u_x^i(x,t),u_{xx}^i(x,t),u) \geq \\ &\geq F_i(x,t,v^i(x,t),v_t^i(x,t),v_x^i(x,t),v_{xx}^i(x,t),v) \quad (i \in \mathbb{N}) \end{aligned} \tag{1}$$

for $(x, t) = (x_1, \ldots, x_n, t) \in D$, where $D \subset \mathbb{R}^n \times (t_0, t_0 + T]$ is one of three relatively arbitrary sets more general than the cylindrical domain $D_0 \times (t_0, t_0 + T] \subset \mathbb{R}^{n+1}$.

The symbol ($w = u$ or v) denotes the mapping:

$$w : \mathbb{N} \times \tilde{D} \ni (i, x, t) \to w^i(x,t) \in \mathbb{R},$$

where $\tilde{D}$ is an arbitrary set such that:

$$\bar{D} \subset \tilde{D} \subset \mathbb{R}^n \times (-\infty, t_0 + T];$$

F_i $(i \in \mathbb{N})$ are functionals of $w, w_x^i = grad_x w^i(x,t)$ and $w_{xx}^i(x,t)$ denote the matrices of second order derivatives with respect to x of $w^i(x,t)$ $(i \in \mathbb{N})$.

We give a theorem on strong maximum principles for problems with inequalities (1) and with nonlocal inequalities together with integrals.

Results obtained are based on those by Besala [1], Brandys [2], [3], Chabrowski [8], Redheffer and Walter [15], Szarski [16], [17], Walter [18], Yoshida [20] and the author [6], [7].

Some infinite and finite, parabolic and hyperbolic systems were considered by Brzychczy [4], [5], Jaruszewska-Walczak [10], Kamont [11], [12], Lakshmikantham and Leea [13], Pudełko [14] and Zabawa [21].

Infinite parabolic systems have physical application. For this purpose please see the publications: [9] by Guiaş and [19] by Wrzosek.

2. Preliminaries

The notation, definitions and assumptions given in this section are applied throughout the paper.

We shall use the following notations:

$$\mathbb{R} = (-\infty, \infty),\ \mathbb{N} = \{1, 2, \ldots\},\ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \ (n \in \mathbb{N}).$$

By ℓ^∞ we denote the Banach space of real sequences $\xi = (\xi^1, \xi^2, \ldots)$ such that:

$$\sup\left\{\left|\xi^j\right| : j = 1, 2, \ldots\right\} < \infty$$

and

$$\|\xi\|_{\ell^\infty} = \sup\left\{\left|\xi^j\right| : j = 1, 2, \ldots\right\}.$$

For $\xi = (\xi^1, \xi^2, \ldots)$, $\eta = (\eta^1, \eta^2, \ldots) \in \ell^\infty$ we write $\xi \le \eta$ in the sense $\xi^i \le \eta^i$ $(i \in \mathbb{N})$.

Let t_0 be a real finite number and let $T \in (0,\infty)$. By $D \subset \{(x,t) : x \in \mathbb{R}^n,\ \ t_0 < t \le t_0 + T\}$ we denote the set of type (*P*) (For the definition of this set see [7]).

For any $t \in [t_0, t_0 + T]$ we define the following sets:

$$S_t = \begin{cases} \operatorname{int}\{x \in \mathbb{R}^n : (x, t_0) \in \bar{D}\} & \text{for } t = t_0, \\ \{x \in \mathbb{R}^n : (x,t) \in D\} & \text{for } t \ne t_0, \end{cases}$$

$$\sigma_t = \begin{cases} \operatorname{int}[\bar{D} \cap (\mathbb{R}^n \times \{t_0\})] & \text{for } t = t_0, \\ D \cap (\mathbb{R}^n \times \{t\}) & \text{for } t \ne t_0. \end{cases}$$

Let $\tilde{D}$ be an arbitrary set such that:

$$\bar{D} \subset \tilde{D} \subset \mathbb{R}^n \times (-\infty, t_0 + T].$$

We introduce the following sets:

$$\partial_p D := \tilde{D} \setminus D \quad \text{and} \quad \Gamma := \partial_p D \setminus \sigma_{t_0}.$$

For an arbitrary fixed point $(\tilde{x}, \tilde{t}) \in D$, we denote by $S^-(\tilde{x}, \tilde{t})$ the set of points $(x,t) \in D$, that can be joined to $(\tilde{x}, \tilde{t})$ by a polygonal line contained in *D* along which the t-coordinate is weakly increasing from (*x*, *t*) to $(\tilde{x}, \tilde{t})$.

Let $Z_\infty(\tilde{D})$ denote the linear space of mappings:

$$w : \mathbb{N} \times \tilde{D} \ni (i, x, t) \to w^i(x,t) \in \mathbb{R},$$

where the functions:

$$w^i : \tilde{D} \ni (x,t) \to w^i(x,t) \in \mathbb{R}$$

are continuous in $\bar{D}$ and

$$\sup\{|w^i(x,t)| : (x,t) \in \tilde{D},\ i \in \mathbb{N}\} < \infty.$$

For $w, \tilde{w} \in Z_\infty(\tilde{D})$ we write $w \le \tilde{w}$ in the sense $w^i \le \tilde{w}^i (i \in \mathbb{N})$.

In the set of mappings *w* belonging to $Z_\infty(\tilde{D})$ we define the functional $[\cdot]_{t,\infty}$ by the formula:

$$[w]_{t,\infty} = \sup\{0, w^i(x, \tilde{t}) : (x, \tilde{t}) \in \tilde{D},\ \ \tilde{t} \le t,\ \ i \in \mathbb{N}\},$$

where $t \le t_0 + T$.

By $Z_\infty^{2,1}(\tilde{D})$ we denote the linear subspace of $Z_\infty(\tilde{D})$. A mapping *w* belongs to $Z(\tilde{D})$ if w_t^i, $w_x^i = (w_{x_1}^i, \ldots, w_{x_n}^i)$, $w_{xx}^i = [w_{x_j x_k}^i]_{n \times n}$ $(i \in \mathbb{N})$ are continuous in *D*.

By $M_{n\times n}(\mathbb{R})$ we denote the space of real square symetric matrices $r = [r_{jk}]_{n\times n}$.

For $r \in M_{n\times n}(\mathbb{R})$ we write $r \geq 0$ if $\sum_{j,k=1}^{n} r_{jk}\lambda_j\lambda_k \geq 0$ for all $(\lambda_1,...,\lambda_n) \in \mathbb{R}^n$.

Let the mappings:

$$F_i : D\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n \times M_{n\times n}(\mathbb{R})\times Z_\infty(\tilde{D}) \ni (x,t,z,p,q,r,w)$$
$$\to F_i(x,t,z,p,q,r,w) \in \mathbb{R} \quad (i \in \mathbb{N})$$

be given and let for an arbitrary function $w \in Z_\infty^{2,1}(\tilde{D})$

$$F_i[x,t,w] := F_i(x,t,w^i(x,t), w_t^i(x,t), w_x^i(x,t), w_{xx}^i(x,t), w),$$
$$(x,t) \in D \ (i \in \mathbb{N}).$$

Two functions $u, v \in Z_\infty^{2,1}(\tilde{D})$ are called solutions of system:

$$F_i[x,t,u] \geq F_i[x,t,v] \quad (i \in \mathbb{N}) \tag{2}$$

in *D*, if they satisfy (2) for $(x, t) \in$ D.

Assumption (L). There are constants $L_i > 0$ (i = 1, 2) such that:

$$F_i(x,t,z,p,q,r,w) - F_i(x,t,z,\tilde{p},q,r,w) \leq L_1(\tilde{p} - p) \quad (i \in \mathbb{N})$$

for all $(x,t) \in D$, $z \in \mathbb{R}$, $p > \tilde{p}$, $q \in \mathbb{R}^n$, $r \in M_{n\times n}(\mathbb{R})$, $w \in Z_\infty(\tilde{D})$
and

$$F_i(x,t,z,p,q,r,w) - F_i(x,t,\tilde{z},\tilde{p},\tilde{q},\tilde{r},\tilde{w})$$

$$\leq L_2(|z-\tilde{z}| + |p-\tilde{p}| + |x|\sum_{j=1}^{n}|q_j - \tilde{q}_j| + |x|^2 \sum_{jk=1}^{n}|r_{jk} - \tilde{r}_{jk}| + [w-\tilde{w}]_{t,\infty}) \quad (i \in \mathbb{N})$$

for all $(x,t) \in D$, $z, \tilde{z} \in \mathbb{R}$, $p, \tilde{p} \in \mathbb{R}$, $q, \tilde{q} \in \mathbb{R}^n$, $r, \tilde{r} \in M_{n\times n}(\mathbb{R})$, $w, \tilde{w} \in Z_\infty(\tilde{D})$.

For every set $A \subset \tilde{D}$ and for each function $w \in Z_\infty(\tilde{D})$ we apply the notation

$$\max_{(x,t)\in A} w(x,t) := \left(\max_{(x,t)\in A} w^1(x,t), \max_{(x,t)\in A} w^2(x,t), ...\right).$$

Let $I = \mathbb{N}$ or I is a finite set of mutually different natural numbers.
Let us define the set:

$$S = \bigcup_{i\in I}\left(\sigma_{T_{2i-1}} \cup \sigma_{T_{2i}}\right),$$

where, in the case if $I = \mathbb{N}$, the following conditions are satisfied:

(i) $t_0 < T_{2i-1} < T_{2i} \leq t_0 + T$ for $i \in I$ and $T_{2i-1} \neq T_{2j-1}, T_{2i} \neq T_{2j}$ for $i, j \in I$, $i \neq j$;

(ii) $T_0 := \inf\{T_{2i-1} : i \in I\} > t_0$;

(iii) $S_t \supset S_{t_0}$ for every $t \in \bigcup_{i\in I}[T_{2i-1}, T_{2i}]$;

(iv) $S_t \supset S_{t_0}$ for every $t \in [T_0, t_0 + T]$,

and in the case if I is a finite set of mutually different natural numbers, the conditions (i), (iii) are satisfied.

An unbounded set D of type (P) is called a set of type ($P_{S\Gamma}$), if:

(a) $S \neq \varnothing$

(b) $\Gamma \cap \bar{\sigma}_{t_0} \neq \varnothing$.

Let S_* denote a non-empty subset of S. We denote the following set:

$$I_* = \left\{ i \in I : \left(\sigma_{T_{2i-1}} \cup \sigma_{T_{2i}} \right) \subset S_* \right\}.$$

A bounded set D of type (P) satisfying condition (a) of the definition of a set of type ($P_{S\Gamma}$) is called a set of type (P_{SB}).

3. Strong maximum principles

Theorem 3.1. Assume that:

1° D is a set of type ($P_{S\Gamma}$) or (P_{SB}).

2° The functions F_i $(i \in \mathbb{N})$ satisfy Assumption (L).

3° $u \in Z_\infty^{2,1}(\tilde{D})$ and the maximum of function u on Γ is attained. Moreover,

$$K^i := \max_{(x,t)\in\Gamma} u^i(x,t) \quad (i \in \mathbb{N}) \tag{3}$$

and $K \in \ell^\infty$ is defined by the formulae

$$K : \mathbb{N} \times \tilde{D} \ni (i,x,t) \to K^i.$$

4° The following inequalities hold:

$$(u^j(x,t_0) - K^j) + \sum_{i\in I_*} h_i(x) \left(\frac{1}{T_{2i} - T_{2i-1}} \int_{T_{2i-1}}^{T_{2i}} u^j(x,\tau)d\tau - K^j \right) \leq 0 \tag{4}$$

for $x \in S_{t_0}$ $(j \in \mathbb{N})$,

where $h_i : S_{t_0} \to \mathbb{R}_\infty$ $(i \in I_*)$ are given functions such that:

$$-1 \leq \sum_{i\in I_*} h_i(x) \leq 0 \quad \text{for} \quad x \in S_{t_0}$$

and additionally if $I_* = \aleph_0$ then the series

$$\sum_{i \in I_*} h_i(x) \frac{1}{T_{2i} - T_{2i-1}} \int_{T_{2i-1}}^{T_{2i}} u^j(x,\tau)d\tau \quad (j \in \mathbb{N})$$

are convergent for $x \in S_{t_0}$.

5° There exists a point $(x^*, t^*) \in \tilde{D}$ such that:

$$u(x^*, t^*) = \max_{(x,t)\in\tilde{D}} u(x,t).$$

Moreover,

$$M^i := u^i(x^*, t^*) \quad (i \in \mathbb{N}) \tag{5}$$

and $M \in \ell^\infty$ is defined by:

$$M : \mathbb{N} \times \tilde{D} \ni (i, x, t) \to M^i.$$

6° u and $\nu = M$ are solutions of system (2) in D.

7° F_i $(i \in \mathbb{N})$ are parabolic with respect to u in D and uniformly parabolic with respect to M in any compact subset of D (see [7]).

Then:

$$\max_{(x,t)\in\tilde{D}} u(x,t) = \max_{(x,t)\in\Gamma} u(x,t). \tag{6}$$

Moreover, if there is a point $(\tilde{x}, \tilde{t}) \in D$ such that:

$$u(\tilde{x}, \tilde{t}) = \max_{(x,t)\in\tilde{D}} u(x,t)$$

then

$$u(x,t) = \max_{(x,t)\in\Gamma} u(x,t) \quad \text{for} \quad (x,t) \in S^-(\tilde{x}, \tilde{t}).$$

Proof. We shall prove Theorem 3.1 for a set of type ($P_{Z\Gamma}$) only, since the proof of this theorem for a set of type (P_{ZB}) is analogous.

We shall argue by contradiction. Suppose that the contrary of (6) holds, i.e.

$$M \neq K$$

Next, (3) and (5) imply inequalities

$$K^i \leq M^i \quad (i \in \mathbb{N}).$$

Consequently,

There is $\ell \in \mathbb{N}$ such that

$$K^\ell < M^\ell. \tag{7}$$

Observe, from assumption 5°, that:

$$\text{There is a point } (x^*, t^*) \in \tilde{D} \text{ such that}$$

$$u(x^*, t^*) = M := \max_{(x,t)\in\tilde{D}} u(x,t). \tag{8}$$

By (8), by assumption 3° and by (7), we have:

$$(x^*, t^*) \in \tilde{D} \setminus \Gamma = D \cup \sigma_{t_0}. \tag{9}$$

Assume that:

$$(x^*, t^*) \in D. \tag{10}$$

From assumption 6° and from (8), we get:

$$\left.\begin{array}{l} u \in Z^{2,1}_{\infty}(\tilde{D}), \\ F_i[x,t,u] \geq F_i[x,t,M] \quad \text{for} \quad (x,t) \in D \ (i \in \mathbb{N}), \\ u(x,t) \leq M \quad \text{for} \quad (x,t) \in \tilde{D}, \\ u(x^*, t^*) = M. \end{array}\right\} \tag{11}$$

The assumption that D is a set of type (P), Assumption (L), relations (10) and (11), and assumption 7° imply, by Theorem 4.1 from [7], the equation:

$$u(x,t) = M \quad \text{for} \quad (x,t) \in S^{-}(x^*, t^*). \tag{12}$$

On the other hand, from the definition of a set of type ($P_{S\Gamma}$) there is a polygonal line $\gamma \subset S^{-}(x^*, t^*)$ such that:

$$\overline{\gamma} \cap \Gamma \neq \varnothing. \tag{13}$$

Since $u^i \in C(\overline{D})$ $(i \in \mathbb{N})$, we have a contradiction of formulae (12) and (13) with formulae (3) and (7). Therefore, $(x^*, t^*) \notin D$ and, consequently, from (9),

$$(x^*, t^*) \in \sigma_{t_0}. \tag{14}$$

Consider now two possible cases:

$$\text{(I)} \quad \sum_{i\in I_*} h_i(x) = 0, \qquad \text{(II)} \quad -1 \leq \sum_{i\in I_*} h_i(x) < 0.$$

In case (I) condition (14) leads, by (7), to a contradiction of (4) with (8). From this contradiction the proof of (6) is complete in case (I).

In case (II), by the definition of sets I and I_*, we must consider the following cases:

(A) I_* is a finite set, i.e., without loss of generality, there is a number $p \in \mathbb{N}$ such that $I_* = \{1, \ldots, p\}$.

(B) $card\ I_* = \aleph_0$.

First we shall consider case (A). By (4) and by the inequalities:

$$u(x^*,t) < u(x^*,t_0) \quad \text{for} \quad t \in \bigcup_{i=1}^{P} [T_{2i-1}, T_{2i}],$$

being a consequence of (8) and (14) and of conditions (a) (i), (a) (iii) of the definition of a set of type ($P_{S\Gamma}$), we have:

$$0 \geq (u^j(x^*,t_0) - K^j) + \sum_{i=1}^{P} h_i(x^*)\left(\frac{1}{T_{2i}-T_{2i-1}}\int_{T_{2i-1}}^{T_{2i}} u^j(x^*,\tau)d\tau - K^j\right) \geq$$

$$\geq (u^j(x^*,t_0) - K^j) + \sum_{i=1}^{P} h_i(x^*)\left(\frac{1}{T_{2i}-T_{2i-1}}\int_{T_{2i-1}}^{T_{2i}} u^j(x^*,t_0)d\tau - K^j\right) =$$

$$= (u^j(x^*,t_0) - K^j)\left(1 + \sum_{i=1}^{P} h_i(x^*)\right) \quad (j \in \mathbb{N}).$$

Hence

$$u(u^*,t_0) \leq K \quad \text{if} \quad 1 + \sum_{i=1}^{P} h_i(x^*) > 0. \tag{15}$$

Then, from (7) and (14), we obtain a contradiction of (15) with (8). Assume now:

$$\sum_{i=1}^{P} h_i(x^*) = -1. \tag{16}$$

By the mean-value integral theorem, we have that for every $j \in \mathbb{N}$ and i $\in$ {1, … p} there is $\tilde{T}_i^j \in [T_{2i-1}, T_{2i}]$ such that:

$$u^j(x^*, \tilde{T}_i^j) = \frac{1}{T_{2i}-T_{2i-1}}\int_{T_{2i-1}}^{T_{2i}} u^j(x^*,\tau)d\tau. \tag{17}$$

Simultaneously, for every $j \in \mathbb{N}$ there is a number $\ell_j \in \{1,\ldots,p\}$ such that:

$$u^j\left(x^*, \tilde{T}_{\ell_j}^j\right) = \max_{i=1,\ldots,p} u^j\left(x^*, \tilde{T}_i^j\right) \tag{18}$$

Consequently, by (16), (18), (17) and by (4), we obtain:

$$u^j\left(x^*,t_0\right) - u^j\left(x^*,\tilde{T}_{\ell_j}^j\right) = \left(u^j\left(x^*,t_0\right) - K^j\right) - \left(u^j\left(x^*,\tilde{T}_{\ell_j}^j\right) - K^j\right) =$$

$$= \left(u^j\left(x^*,t_0\right) - K^j\right) + \sum_{i=1}^{P} h_i(x^*)\left(u^j\left(x^*,\tilde{T}_{\ell_j}^j\right) - K^j\right) \leq$$

$$\leq \left(u^j\left(x^*,t_0\right)-K^j\right)+\sum_{i=1}^{P} h_i(x^*)\left(u^j\left(x^*,\tilde{T}_i^j\right)-K^j\right)=$$

$$=\left(u^j\left(x^*,t_0\right)-K^j\right)+\sum_{i=1}^{P} h_i(x^*)\left(\frac{1}{T_{2i}-T_{2i-1}}\int_{T_{2i-1}}^{T_{2i}} u^j(x^*,\tau)d\tau-K^j\right)\leq 0 \quad (j\in\mathbb{N}).$$

Hence

$$u^j\left(x^*,t_0\right)\leq u^j\left(x^*,\tilde{T}_{\ell_j}^j\right) \quad (j\in\mathbb{N}) \quad \text{if} \quad \sum_{i=1}^{P} h_i(x^*)=-1. \tag{19}$$

Since, by (a) (i) of the definition of a set of type ($P_{S\Gamma}$), $\tilde{T}_{\ell_j}^j > t_0$ $(j\in\mathbb{N})$, we get from (14) that (19) is at a contradiction with (8). This completes the proof of (6) if I_* is a finite set.

It remains to investigate case (B). Analogously to the proof of (6) in case (A), by assumption 4° and by the inequality:

$$u\left(x^*,t\right)<u\left(x^*,t_0\right) \quad \text{for} \quad t\in\bigcup_{i\in I_*}[T_{2i-1},T_{2i}],$$

being a consequence of (8), (14), and of (a) (i), (a) (iii) of the definition of a set of type ($P_{S\Gamma}$) we have:

$$0\geq (u^j(x^*,t_0)-K^j)+\sum_{i\in I_*} h_i(x^*)\left(\frac{1}{T_{2i}-T_{2i-1}}\int_{T_{2i-1}}^{T_{2i}} u^j(x^*,\tau)d\tau-K^j\right)\geq$$

$$\geq (u^j(x^*,t_0)-K^j)+\sum_{i\in I_*} h_i(x^*)\left(\frac{1}{T_{2i}-T_{2i-1}}\int_{T_{2i-1}}^{T_{2i}} u^j(x^*,t_0)d\tau-K^j\right)=$$

$$=(u^j(x^*,t_0)-K^j)(1+\sum_{i\in I_*} h_i(x^*)) \quad (j\in\mathbb{N}).$$

Hence

$$u(x^*,t_0)\leq K \quad \text{if} \quad 1+\sum_{i\in I_*} h_i(x^*)>0. \tag{20}$$

Then from (7) and (14), we obtain a contradiction of (20) with (8). Assume now:

$$\sum_{i\in I_*} h_i(x^*)=-1. \tag{21}$$

By the mean-value integral theorem we have that for every $j\in\mathbb{N}$ and $i \in I_*$ there is $\tilde{T}_i^j\in\left[T_{2i-1},T_{2i}\right]$ such that:

$$u^j(x^*,\tilde{T}_i^j)=\frac{1}{T_{2i}-T_{2i-1}}\int_{T_{2i-1}}^{T_{2i}} u^j(x^*,\tau)d\tau. \tag{22}$$

Let

$$\tilde{T}_*^j := \inf_{i \in I_*} \tilde{T}_i^j \quad (j \in \mathbb{N}). \tag{23}$$

Since $u^i \in C(\overline{D})$ $(i \in \mathbb{N})$ and since, by (14) and by (a) (iv), (a) (ii) of the definition of a set of type $(P_{S\Gamma})$, $x^* \in S_t$ for every $t \in [T_0, t_0 + T]$ if $I = \mathbb{N}$, it follows from (23) that for every $j \in \mathbb{N}$ there is $\hat{t}_j \in \left[\tilde{T}_*^j, t_0 + T\right]$ such that:

$$u^j(x^*, \hat{t}_j) = \max_{t \in \left[\tilde{T}_*^j, t_0 + T\right]} u^j(x^*, t). \tag{24}$$

Consequently, by (21), (24), (22) and by assumption 4°, we obtain:

$$\begin{aligned} u^j(x^*, t_0) - u^j(x^*, \hat{t}_j) &= (u^j(x^*, t_0) - K^j) - (u^j(x^*, \hat{t}_j) - K^j) = \\ &= (u^j(x^*, t_0) - K^j) + \sum_{i \in I_*} h_i(x^*)(u^j(x^*, \hat{t}_j) - K^j) \le \\ &\le (u^j(x^*, t_0) - K^j) + \sum_{i \in I_*} h_i(x^*)(u^j(x^*, \tilde{T}_i^j) - K^j) = \\ &= (u^j(x^*, t_0) - K^j) + \sum_{i \in I_*} h_i(x^*)\left(\frac{1}{T_{2i} - T_{2i-1}} \int_{T_{2i-1}}^{T_{2i}} u^j(x^*, \tau)d\tau - K^j\right) \le 0 \quad (j \in \mathbb{N}). \end{aligned}$$

Hence

$$u^j(x^*, t_0) \le u^j(x^*, \hat{t}_j) \quad \text{if} \quad \sum_{i \in I_*} h_i(x^*) = -1. \tag{25}$$

Since, by (a) (ii) of the definition of a set of type $(P_{S\Gamma})$, $t_j > t_0$ $(j \in \mathbb{N})$, we get from (14) that (25) is at a contradiction with (8). This completes the proof of equality (6).

The second part of the thesis of Theorem 3.1 is a consequence of (6) and of Theorem 4.1 from [7]. Therefore, the proof of Theorem 3.1 is complete.

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