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ON CONVERGENCE ESTIMATES FOR THE FDM WITH IRREGULAR MESH

SZACOWANIE ZBIEŻNOŚCI MRS Z NIEREGULARNĄ SIATKĄ WĘZŁÓW

A b s t r a c t

An error estimation technique for solutions of parabolic boundary-value problems, obtained by finite difference method with irregular meshes in the local formulation, is presented. Explicit and implicit schemes are used. Convergence tests illustrate the method.

Keywords: *finite difference method, irregular mesh*

S t r e s z c z e n i e

W artykule przedstawiono sposób szacowania *a priori* błędu metody różnic skończonych z nieregularną siatką węzłów w ujęciu lokalnym dla różnych problemów brzegowych typu parabolicznego. Wykorzystano jawne i niejawne schematy różnicowe. Zamieszczono wyniki testów zbieżności.

Słowa kluczowe: *metoda różnic skończonych, nieregularna siatka węzłów*

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1. Introduction

The Finite Difference Method (FDM) with irregular meshes, used first in the half of the previous century by R. H. Mac Neal [15] and continued later by many authors was applied mainly to elliptic boundary-value problems (BVP). The method was developed in two directions: in the first one, called "balance method", a mesh of triangles or rectangles was used and difference formulas were obtained using the Gaussian formula and linear interpolation ([5] and references therein). In the second approach, similar to the classical one, only nodal values of functions were used and the difference scheme was obtained by the Taylor expansion ([13, 14, 19] and references therein).

Mathematical proofs were done mainly for the first version [2, 3, 5, 10, 18]. For the local formulation on a regular mesh proofs of convergence are well-known. Not so many of them have been done for the FDM on irregular meshes [1, 4, 9, 17].

For parabolic BVP the classical FDM with regular meshes was commonly used but there are few papers concerning FDM with irregular meshes. T. Liszka and J. Orkisz in [13, 14] present an approximate solution of the heat equation obtained by "irregular" formulas on a regular mesh. An efficient stability condition was obtained this way. A proof of the convergence for one dimensional space variable was obtained by Li Ronghua [18]. FDM with rectangular meshes was considered by M. Malec and M. Rosati [16].

An overview of the FDM with irregular mesh and its applications was done by F. Ihlenburg [6], J. Orkisz [20] and J. Krok [12], who published many papers in this subject, e.g. [11].

The purpose of this paper is to present convergence analysis for solutions of parabolic BVP obtained by the FDM with irregular meshes in the local formulation in an explicit and implicit form. The presented proof is based on a discrete maximum principle (Theorems 3.1 and 3.2) extended in an earlier paper ([7]). Theoretical results are confirmed by convergence tests.

2. Statement of the problem

A cylindrical, bounded domain

$D=(0,T)\times\Omega\subset\Re^3$ with a boundary $\partial D=\partial D_0\cup\partial D_T\cup\partial D_1\cup\partial D_2$ is considered. The symbols used above denote

$$\partial D_0=\{0\}\times\bar{\Omega},\quad \partial D_T=\{T\}\times\Omega,\quad \partial D_1\cup\partial D_2=\partial D\setminus(\partial D_0\cup\partial D_T)=(0,T]\times\partial\Omega,$$
$$\partial D_1\cap\partial D_2=\emptyset,$$

where $\bar{\Omega}$, $\partial\Omega$ are closure and boundary of Ω correspondingly, ∂D_1, ∂D_2 will be defined by boundary conditions and

$$\partial D_B=\partial D_0\cup\partial D_1,\qquad \bar{D}=D\cup\partial D.$$

Let a mesh of nodes $x_i=(x^1{}_i,x^2{}_i)\in\Omega,\ i\in I$ be given with

$I_{\mathrm{int}}=\{i:x_i\in\Omega\},\quad I_{bd}=\{i:x_i\in\partial\Omega\}$ – indices of internal and boundary nodes.

The time interval $[0,T]$ is divided into M equal time steps

$$0 = t_0 < t_1 < \ldots < t_M = T, \quad k = t_n - t_{n-1} = \text{const}, \quad 1 \le n \le M$$

and (t_n, x_i), $0 \le n \le M$, $i \in I$ are the mesh of nodes in $\bar{D}$. For each node $(t_n, x_i) \in \bar{D}$ a set of indices j of auxiliary nodes connected with this node ("star of the node"), $G(n,i)$ is defined ($i \notin G(n,i)$). We denote further

$$h(n,i) = \sup(|(t_n, x_j) - (t_n, x_i)|,\ j \in G(n,i)\}, \quad h = \{\max h(n,i),\ n \in \{1, \ldots, M\},\ i \in I\},$$

$$u,_t = \frac{\partial u}{\partial t}, \quad u,_i = \frac{\partial u}{\partial x^i}, \quad u,_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}, \quad \Delta u = u,_{11} + u,_{22}, \quad u,_n = \frac{\partial u}{\partial n}.$$

Moreover we assume that

$$\forall (t_n, x_i) \in \partial D_2, \quad \forall j \in G(n,i), \quad (t_n, x_j) \notin \partial D_1 \cup \partial D_2,$$

i.e. each star of a node which belongs to ∂D_2 consists of internal nodes only. In the domain D the following boundary-value problem is considered

$$Lu(t,x) = u,_t(t,x) - \left[\sum_{i,j=1}^{2} a_{ij}(t,x) \cdot u,_{ij}(t,x) + \sum_{i=1}^{2} b_i(t,x) \cdot u,_i(t,x) + c(t,x) \cdot u(t,x) \right] = f(t,x) \quad (1)$$

for $(t,x) \in D$

$$Bu(t,x) = \sum_{i=1}^{2} d_i(t,x) \cdot u,_i(t,x) + e(t,x) \cdot u(t,x) = g_2(t,x) \quad (2)$$

for $(t,x) \in \partial D_2$, and

$$u(t,x) = g_1(t,x) \quad (3)$$

for $(t,x) \in \partial D_B$.

The problem (1)–(3) is replaced by the FDM equations for the discrete solution U

$$\begin{aligned} L_h^i(U) = {} & k^{-1}\left[U(t_n, x_i) - U(t_{n-1}, x_i)\right] \\ & - \sum_{j \in G(n,i)} \left\{ \left[\alpha_j(n,i) h^{-2} + \beta_j(n,i) h^{-1}\right] \cdot \left[U(t_n, x_j) - U(t_n, x_i)\right] \right\} \\ & - \gamma(n,i) U(t_n, x_i) = f(t_n, x_i) \end{aligned} \quad (4)$$

(implicit scheme), or

$$\begin{aligned} L_h^e(U) = {} & k^{-1}\left[U(t_{n+1}, x_i) - U(t_n, x_i)\right] \\ & - \sum_{j \in G(n,i)} \left\{ \left[\alpha_j(n,i) h^{-2} + \beta_j(n,i) h^{-1}\right] \cdot \left[U(t_n, x_j) - U(t_n, x_i)\right] \right\} \\ & - \gamma(n,i) U(t_n, x_i) = f(t_n, x_i) \end{aligned} \quad (5)$$

(explicit scheme), for $(t_n, x_i) \in D \cup \partial D_T$

$$B_h(U) = h^{-1} \sum_{j \in G(n,i)} \delta_j(n,i) \cdot \left[U(t_n, x_i) - U(t_n, x_j) \right] + \varepsilon(n,i) U(t_n, x_i) = g_2(t_n, x_i) \tag{6}$$

for $(t_n, x_i) \in \partial D_2$

$$U(t_n, x_i) = g_1(t_n, x_i) \tag{7}$$

for $(t_n, x_i) \in \partial D_B$.

3. Maximum principle

Theorem 3.1 ([7]). Let the implicit difference system (4), (6), (7) fulfils at each node (t_n, x_i) the following assumptions

$$\alpha_j(n,i), \delta_j(n,i) > 0, \quad \alpha_j(n,i) h^{-1} + \beta_j(n,i) > 0 \quad \forall j \in G(n,i), \quad \varepsilon(n,i) \geq 0, \quad \gamma(n,i) \leq 0, \tag{8}$$

$$f(t_n, x_i) \leq 0, \quad g_1(t_n, x_i) \leq 0 \quad g_2(t_n, x_i) \leq 0, \tag{9}$$

then

$$U(t_n, x_i) \leq 0 \tag{10}$$

at each node $(t_n, x_i) \in \bar{D}$.

Theorem 3.2 ([7]). Let the explicit difference system (5)–(7) fulfils at each node (t_n, x_i) assumptions (8), (9) and

$$0 < k \leq \min \left\{ \sum_{j \in G(n,i)} \left\{ \left[\alpha_j(n,i) h^{-2} + \beta_j(n,i) h^{-1} \right] - \gamma(n,i) \right\}^{-1}, \quad n \in \{0, \ldots, M\}, i \in I \right\} \tag{11}$$

then (10) holds at each node.

Sketch of the proof of Theorem 3.1. Suppose, on the contrary, that there is a node (t_n, x_k) such that

$$U(t_n, x_k) > 0. \tag{12}$$

(9) implies that

$$(t_n, x_k) \notin \partial D_B. \tag{13}$$

There are two possible cases

1. $(t_n, x_k) \in \partial D_2$, then

$$0 < U(t_n, x_k) \cdot \left[\sum_{j \in G(n,k)} \delta_j + h\varepsilon \right] = \sum_{j \in G(n,k)} \delta_j U(t_n, x_j) + h g_2(t_n, x_k)$$
$$\leq \sum_{j \in G(n,k)} \delta_j \max\{U(t_n, x_p),\ p \in I_{\text{int}}\}$$

and there is a node $(t_n, x_r) \in D$ at which

$$0 < U(t_n, x_k) < U(t_n, x_r),$$

what implies the second case.

2. $(t_n, x_k) \in D.$ (12), (13) imply then an existence of a $m = m(n) \in I_{\text{int}}$ such that

$$U(t_n, x_m) = \max\{U(t_n, x_i),\ i \in I\} > 0.$$

For this node we obtain

$$U(t_{n-1}, x_m) = -k \cdot \sum_{j \in G(n,m)} \left\{ \left[\alpha_j h^{-2} + \beta_j h^{-1} \right] \cdot \left[U(t_n, x_j) - U(t_n, x_m) \right] \right\} + U(t_n, x_m)$$
$$- kf(t_n, x_m) \geq -k \cdot \sum_{j \in G(n,m)} \left[\alpha_j h^{-2} + \beta_j h^{-1} \right] \cdot U(t_n, x_j)$$
$$+ \left[1 + k \cdot \sum_{j \in G(n,m)} \left[\alpha_j h^{-2} + \beta_j h^{-1} \right] \cdot U(t_n, x_m) \right] \geq U(t_n, x_m) > 0.$$

We can then write

$$\max\{U(t_{n-k}, x_p),\ p \in I_{\text{int}}\} \geq \max\{U(t_n, x_p),\ p \in I_{\text{int}}\} > 0, \quad k = 1, \ldots, n,$$

but this implies

$$0 < \max\{U(0, x_p), (t_n, x_p) \in \partial D_B\},$$

which contradicts (9).

Proof of the Theorem 3.2 is similar (cf. [7]).

4. Convergence of the method

Suppose the following conditions hold:

A1. The BVP (1)–(3) has the unique solution $u \in C^3(D) \cap C^2(\bar{D})$ and

$$\sup\left\{ |D^\alpha u(t,x)|, (t,x) \in D, |\alpha| \leq 3 \right\} = M_1 < \infty,$$

$$\sup\left\{ |D^\alpha u(t,x)|, (t,x) \in \bar{D}, |\alpha| \leq 2 \right\} = M_2 < \infty.$$

A2. For any polynomial V of the form

$$V(t,x) = a_{11}x^1x^1 + a_{12}x^1x^2 + a_{22}x^2x^2 + a_{10}x^1 + a_{01}x^2 + a_t t + a_{00},$$

if the implicit scheme is used, the equality

$$LV(t_n, x_i) = L_h^i V(t_n, x_i), \quad \forall (t_n, x_i), \tag{14a}$$

or, in the other case

$$LV(t_n, x_i) = L_h^e V(t_n, x_i), \quad \forall (t_n, x_i) \tag{14b}$$

holds.

A3. Inequalities (8) are fulfilled.

A4. There is $m > 0$ independent of k, h, n such that

$$\sum_{j\in G(n,i)} \left(\alpha_j + \beta_j\right) - \gamma \le m, \quad \forall i \in I_{\text{int}}, \qquad \sum_{j\in G(n,i)} \delta_j \le m, \quad \forall i \in I_{bd}.$$

A5. If $\partial D_2 \ne \emptyset$ then there are constants $c_1, M_3, M_4 > 0$ and a function $v \in C^3(D) \cap C^2(\bar{D})$ such that

$$Lv \ge c_1 \quad \text{in} \quad D \cup D_T,$$

$$Bv \ge c_1 \quad \text{on} \quad \partial D_2,$$

$$v \ge 0 \quad \text{on} \quad \partial D_B,$$

$$\sup\left\{| D^\alpha v(t,x) |, (t,x) \in D, |\alpha| \le 3\right\} = M_3 < \infty,$$

$$\sup\left\{| D^\alpha v(t,x) |, (t,x) \in \bar{D}, |\alpha| \le 2\right\} = M_4 < \infty.$$

A6. The difference problem (4), (6), (7) or, if explicit scheme is used, (5)–(7) has the unique solution *U*.

Theorem 4.1. If Assumptions A1–A6 are fulfilled (with (14a)), then the discrete solution *U* obtained by the implicit method, converges to the exact solution *u* and there is a constant $C > 0$ independent of *k* and *h* such that

$$\sup\left\{| U(t_n, x_i) - u(t_n, x_i) |, (t_n, x_i) \in \bar{D}\right\} \le C(k+h). \tag{15}$$

Theorem 4.2. If Assumptions A1–A6 are fulfilled (with (14b)), inequality (11) holds and *U* is the discrete solution obtained by the explicit method, then the assertion of Theorem 4.1. holds.

Remark 1. Assumptions A1 and A5 may be replaced by a much stronger assumption: "For any right hand sides of the equations (1)–(3) regular enough, there is a unique solution of these equations with bounded second and third derivative". This condition is forced here to the searched exact solution and to one solution of the problem with positive right hand sides only. Then Assumption A5 does not limit in fact applications of Theorems 4.1 and 4.2.

Remark 2. For the heat conduction problem as a typical phenomenon described by a partial differential equation of parabolic type the solution *u* of the considered problem or the function *v* in Assumption A5 are the temperature distributions, *Lu* and *Lv* are distributions of heat sources, *Bu*, *Bv* represent heat flows through the boundary. Assumption A5 says that for positive heat sources and flow the temperature may be positive with limited second and third derivatives, i.e. there is no singularities e.g. at the contact of the Dirichlet and Neumann boundary conditions.

Proof of Theorems 4.1 and 4.2.

The solution u of (1)–(3) can be expanded in Taylor series

$$u(t,x)=u(t_n,x_i)+u,_t(t_n,x_i)(t-t_n)+\sum_{j=1}^{2}u,_j(t_n,x_i)(x^j-x_i^j)+0,5u,_{tt}(t_n,x_i)(t-t_n)^2$$

$$+\sum_{j=1}^{2}u,_{tj}(t_n,x_i)(t-t_n)(x^j-x_i^j)+0,5\sum_{k,j=1}^{2}u,_{kj}(t_n,x_i)(x^j-x_i^j)(x^k-x_i^k)$$

$$+\frac{1}{6}u,_{ttt}(t_n,x_i)(t-t_n)^3+0,5\sum_{j=1}^{2}u,_{ttj}(t',x')(t-t_n)^2(x^j-x_i^j)$$

$$+0,5\sum_{k,j=1}^{2}u,_{tkj}(t',x')(t-t_n)(x^j-x_i^j)(x^k-x_i^k)$$

$$+\frac{1}{6}\sum_{l,k,j=1}^{2}u,_{lkj}(t',x')(x^j-x_i^j)(x^k-x_i^k)(x^l-x_i^l)=u(t_n,x_i)+u,_t(t_n,x_i)(t-t_n)$$

$$+\sum_{j=1}^{2}u,_j(t_n,x_i)(x^j-x_i^j)+0,5\sum_{k,j=1}^{2}u,_{kj}(t_n,x_i)(x^j-x_i^j)(x^k-x_i^k)+R(t,x).$$

At the point (t_n,x_i) we have

$$(L-L_h)u(t_n,x_i)=L_h(U-u)(t_n,x_i)=f(t_n,x_i)$$

$$-L_h\Bigg(u(t_n,x_i)+u,_t(t_n,x_i)(t-t_n)+\sum_{j=1}^{2}u,_j(t_n,x_i)(x^j-x_i^j)$$

$$+0,5\sum_{k,j=1}^{2}u,_{kj}(t_n,x_i)(x^j-x_i^j)(x^k-x_i^k)\Bigg)-L_h(R)$$

$$=f(t_n,x_i)-f(t_n,x_i)+O(k+h)=O(k+h).$$

Similarly

$$u(t,x)=u(t_n,x_i)+u,_t(t_n,x_i)(t-t_n)+\sum_{j=1}^{2}u,_j(t_n,x_i)(x^j-x_i^j)+0,5u,_{tt}(t',x')(t-t_n)^2$$

$$+\sum_{j=1}^{2}u,_{tj}(t',x')(t-t_n)(x^j-x_i^j)+0,5\sum_{k,j=1}^{2}u,_{kj}(t',x')(x^j-x_i^j)(x^k-x_i^k)$$

$$=u(t_n,x_i)+\sum_{j=1}^{2}u,_j(t_n,x_i)(x^j-x_i^j)+P(t,x).$$

$$(B-B_h)u(t_n,x_i)=B_h(U-u)(t_n,x_i)=g_2(t_n,x_i)$$
$$-B_h\left(u(t_n,x_i)+\sum_{j=1}^{2}u,_j(t_n,x_i)(x^j-x_i^j)\right)-B_h(P)$$
$$=g_2(t_n,x_i)-g_2(t_n,x_i)+O(h)=O(h)$$

The above formulas can be rewritten as

$$|L_h(U-u)|\le c_2(k+h),\qquad |B_h(U-u)|\le c_2h. \tag{16}$$

For the function v defined in Assumption A5, or, if $\partial D_2=\emptyset$, $v(t,x)=c_1t$, the following assumptions hold

$$(L-L_h)v(t_n,x_i)=O(k+h),\qquad (B-B_h)v(t_n,x_i)=O(h),$$

then for k, h small enough

$$(L-L_h)v(t_n,x_i)\le 0,5c_1\quad \forall(t_n,x_i)\in D\cup\partial D_T,$$
$$(B-B_h)v(t_n,x_i)\le 0,5c_1\quad \forall(t_n,x_i)\in \partial D_2, \tag{17}$$

therefore

$$L_hv(t_n,x_i)\ge 0,5c_1,\qquad B_hv(t_n,x_i)\ge 0,5c_1. \tag{18}$$

Let $v_h=2(k+h)vc_2c_1^{-1}$. Then, by (16)–(18) and A5

$$L_h(U-u-v_h)(t_n,x_i)\le 0\quad\text{in}\quad D\cup\partial D_T,$$
$$B_h(U-u-v_h)(t_n,x_i)\le 0\quad\text{on}\quad \partial D_2,$$
$$(U-u-v_h)(t_n,x_i)\le 0\quad\text{on}\quad \partial D_B.$$

The same inequalities imply

$$L_h(u-U-v_h)(t_n,x_i)\le 0\quad\text{in}\quad D\cup\partial D_T,$$
$$B_h(u-U-v_h)(t_n,x_i)\le 0\quad\text{on}\quad \partial D_2,$$
$$(u-U-v_h)(t_n,x_i)\le 0\quad\text{on}\quad \partial D_B,$$

therefore, by Theorems 3.1 and 3.2, we have

$$(U-u-v_h)(t_n,x_i)\le 0\quad\text{in}\quad D,$$
$$(u-U-v_h)(t_n,x_i)\le 0\quad\text{in}\quad D\quad\text{and}$$
$$|(u-U)(t_n,x_i)|\le v_h(t_n,x_i)\le c(k+h)\quad \forall(t_n,x_i)\in D.$$

5. Numerical tests

The cylindrical domain

$$D = \{(t, x^1, x^2) : 0 < t < 1, r^2 = (x^1)^2 + (x^2)^2 < 1\}$$

with equal time-steps and irregular mesh of nodes at each time was considered. The following boundary-value problems were defined:
BVP1. The Dirichlet BVP

$$u,_t - \Delta u = f \text{ in } D, \quad u = g_1 \text{ on } \partial D_B, \quad \partial D_2 = \emptyset.$$

BVP2. The Neumann BVP

$$u,_t - \Delta u + 10u = f \text{ in } D, \quad u = g_1 \text{ on } \partial D_0, \quad u,_n = g_2 \text{ on } \partial D_2, \quad \partial D_1 = \emptyset.$$

BVP3. The Robin BVP

$$u,_t - \Delta u = f \text{ in } D, \quad u = g_1 \text{ on } \partial D_0, \quad u,_n + u = g_2 \text{ on } \partial D_2, \quad \partial D_1 = \emptyset.$$

BVP4. The mixed BVP

$$u,_t - \Delta u = f \text{ in } D, \quad u = g_1 \text{ on } \partial D_B, \quad u,_n = g_2 \text{ on } \partial D_2 = \{(t, x) : r = 1, x^1 > 0\}.$$

Values of f, g_1, g_2 were taken as corresponding values for exact solutions in four examples:

1. $u(t, r) = 1 - t \cdot r^4$,
2. $u(t, r) = \cos(0,25\pi tr)$,
3. $u(t, r) = \sin(0,3\pi t^2 r)$,
4. $u(t, r) = e^{-tr^2}$.

Each example was solved with the explicit and implicit scheme. The expression

$$e = \sup\{|u(t_i, x_j) - U(t_i, x_j)|, (t_i, x_j) \in \bar{D}\}$$

was assumed to be the error of the method.

An example of an irregular mesh in the circle $r^2 \leq 1$ is shown in Fig. 1. The graphs of the approximate and exact solution on the plane $x^2 = 0$ for the Boundary Value Problem 1, Example 3, explicit scheme, are given in Figs. 2 and 3 (the both solutions are axisymmetric). Rates of convergence in all Boundary Value Problems and all examples are presented in Figs. 4–11. The *x* and *y* axis represent logarithms of *h* and of *e*. Rates of convergence for considered problems are assembled in the Table 1.

Table 1

Rates of convergence

	BVP 1		BVP 2		BVP 3		BVP 4	
Scheme	explicit	implicit	explicit	implicit	explicit	implicit	explicit	implicit
Solution 1	2,073	2,074	1,182	1,182	0,927	0,936	0,989	0,999
Solution 2	1,979	2,141	1,377	1,378	1,065	1,085	1,133	1,155
Solution 3	1,976	2,081	1,502	1,492	1,104	1,104	1,147	1,147
Solution 4	2,211	2,207	1,074	1,076	0,959	1,037	1,031	1,086

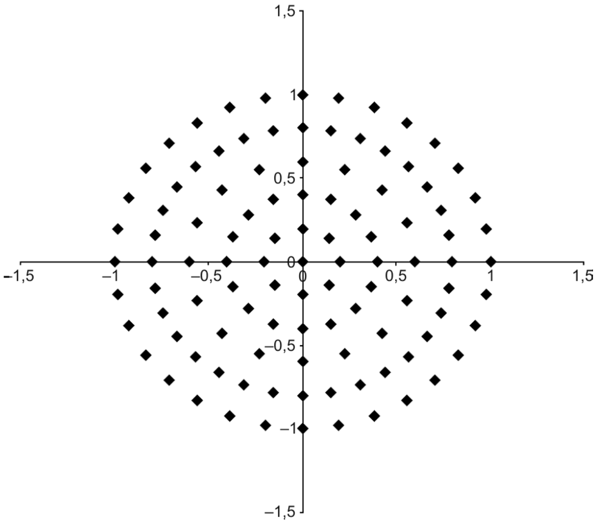

Fig. 1. Irregular mesh of nodes
Rys. 1. Nieregularna siatka węzłów

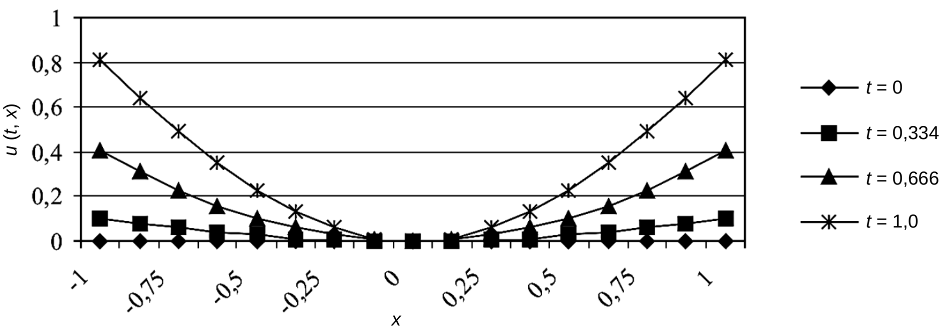

Fig. 2. Approximate solution for $y = 0$
Rys. 2. Rozwiązanie przybliżone dla $y = 0$

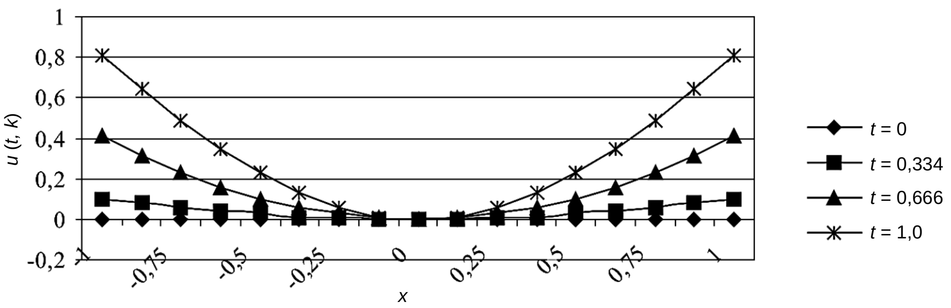

Fig. 3. Exact solution for $y = 0$
Rys. 3. Rozwiązanie dokładne dla $y = 0$

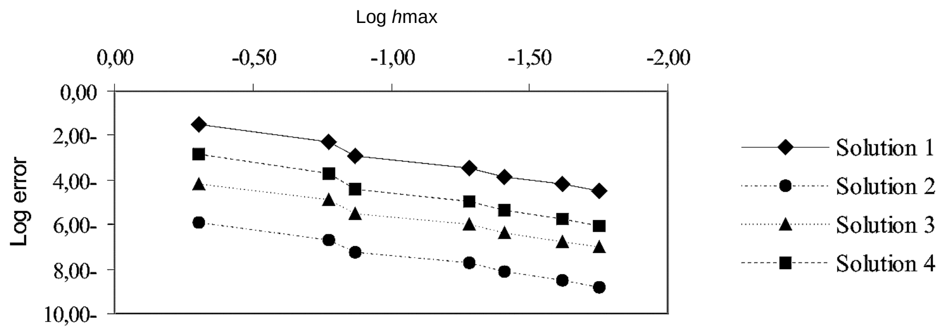

Fig. 4. Convergence test: Boundary Value Problem 1, explicit scheme
Rys. 4. Test zbieżności: problem brzegowy nr 1, schemat jawny

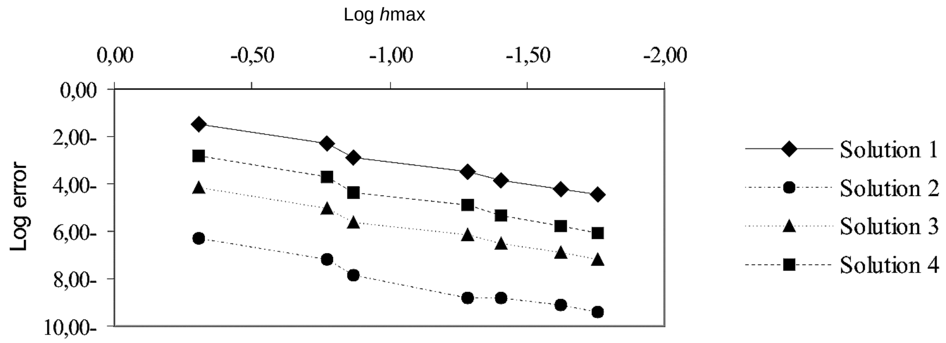

Fig. 5. Convergence test: Boundary Value Problem 1, implicit scheme
Rys. 5. Test zbieżności: problem brzegowy nr 1, schemat niejawny

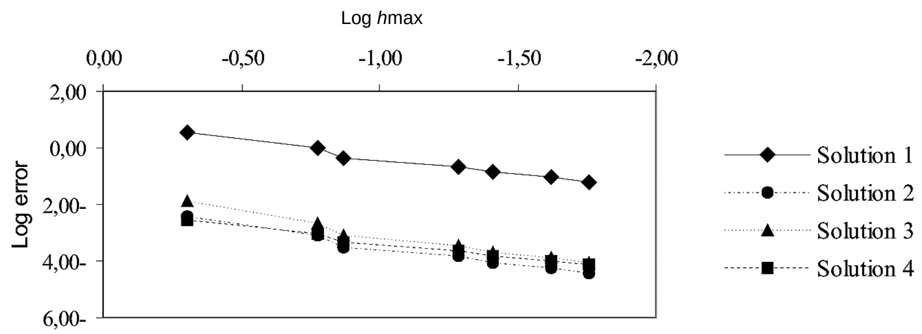

Fig. 6. Convergence test: Boundary Value Problem 2, explicit scheme
Rys. 6. Test zbieżności: problem brzegowy nr 2, schemat jawny

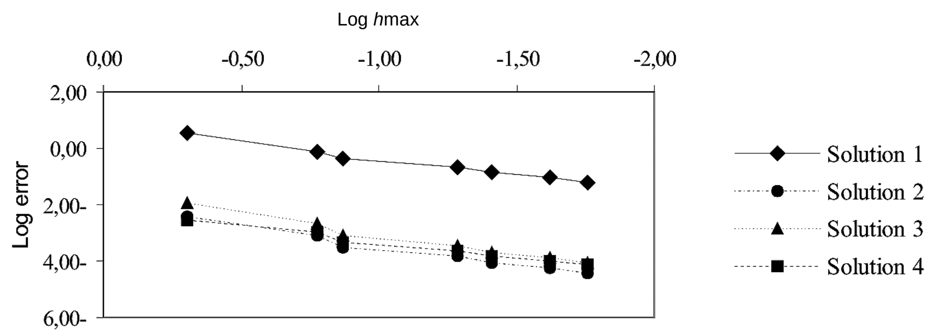

Fig. 7. Convergence test: Boundary Value Problem 2, implicit scheme
Rys. 7. Test zbieżności: problem brzegowy nr 2, schemat niejawny

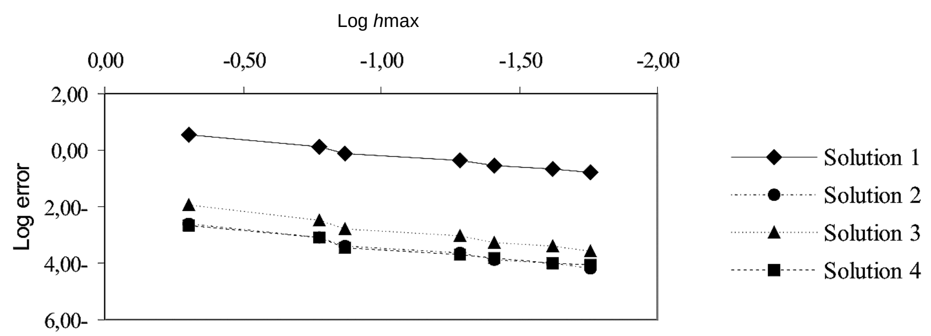

Fig. 8. Convergence test: Boundary Value Problem 3, explicit scheme
Rys. 8. Test zbieżności: problem brzegowy nr 3, schemat jawny

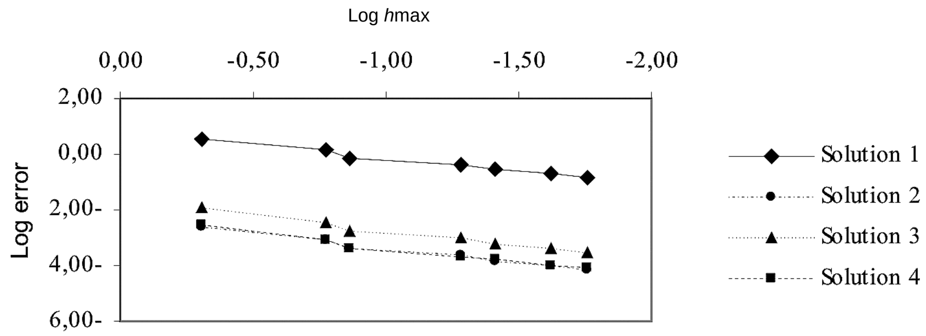

Fig. 9. Convergence test: Boundary Value Problem 3, implicit scheme
Rys. 9. Test zbieżności: problem brzegowy nr 3, schemat niejawny

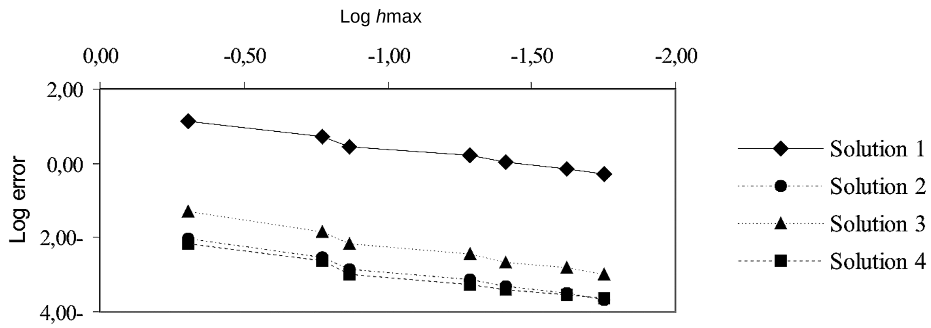

Fig. 10. Convergence test: Boundary Value Problem 4, explicit scheme
Rys. 10. Test zbieżności: problem brzegowy nr 4, schemat jawny

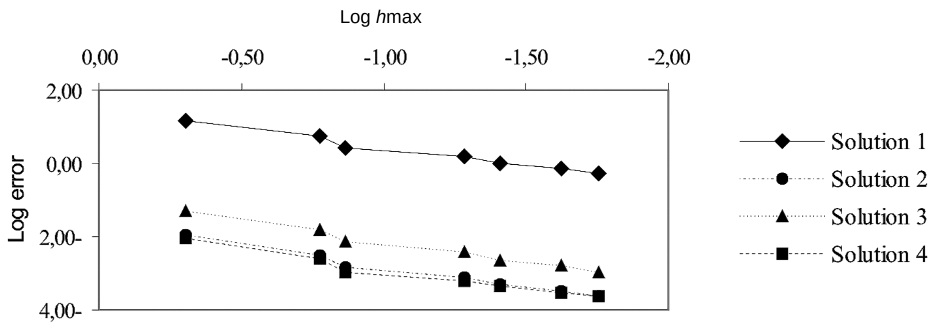

Fig. 11. Convergence test: Boundary Value Problem 4, implicit scheme
Rys. 11. Test zbieżności: problem brzegowy nr 4, schemat niejawny

6. General remarks

The presented numerical examples converge to exact solutions with the rate $O(h^2)$ (BVP1) or $O(h)$ (BVP 2–4), for the simple boundary condition approximation. When the approximation along the boundary is of the second order, the obtained convergence rate for the BVP2 is $O(h^2)$. It suggest that for given assumptions, convergence estimate (16) is optimal. The problem of modifying (16) for the Dirichlet BVP and for the second order boundary approximation remains open. The presented examples show the importance of the accurate approximation of the boundary conditions containing derivatives. For the given end-time t = 1 convergence of the both FD methods is theoretically and numerically confirmed.

The paper does not present stability tests. Theoretical estimates however include the stability condition (11). It coincides with the classical one on a regular mesh. In a special case better results were obtained by Liszka and Orkisz ([13, 14]).

This paper may be extended on another difference schemes, for example Crank–Nicholson schemes, schemes with excessive nodes [14] or schemes obtained by the balance method [5]. Application of the presented method to other boundary value problems may be investigated, too.

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