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A LINEAR ABSTRACT INITIAL VALUE PROBLEM WITH BESOV FUNCTION

ABSTRAKCYJNY LINIOWY PROBLEM POCZĄTKOWY Z FUNKCJĄ BESOVA

A b s t r a c t

This article is devoted to the investigation of the abstract linear initial value problem: $(*)\begin{cases}\frac{du}{dt}(t)+A(t)u(t)=f(t)\\ u(s)=x\end{cases}$ in a Banach space in a parabolic case with Besov function *f*. We give sufficient condition for existence and uniqueness of the solution of the problem $(*)$ which may have weak singularity at the origin.

Keywords: *Besov space*, *semigroup with singularity*

S t r e s z c z e n i e

Niniejszy artykuł dotyczy abstrakcyjnego problemu początkowego: $(*)\begin{cases}\frac{du}{dt}(t)+A(t)u(t)=f(t)\\ u(s)=x\end{cases}$ w przestrzeni Banacha z funkcją Besova *f*. Podane są warunki wystarczające na istnienie i jednoznaczność rozwiązania problemu $(*)$ ze słabą osobliwością w punkcie początkowym.

Słowa kluczowe: *przestrzeń Besova, półgrupa z osobliwością*

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1. Introduction

Let X be a Banach space, and let $\{A(t)\}_{t\in[0,T]}$ be a family of closed densely defined linear operators from X to X with domains $D = D(A(t))$ independent on t. We consider the linear abstract initial value problem

$$\begin{cases} \frac{du}{dt}(t) + A(t)u(t) = f(t), & t \in (s,T], \\ u(s) = x, & s \in [0,T),\ x \in X, \end{cases} \tag{1}$$

where $f:[0,T] \to X$ is a Besov function. We prove a theorem on the existence and uniqueness of the solution of problem (1).

Definition 1. *A function* $u:[s,T] \to X$ *is said to be a classical solution of the problem* (1) *if*

(i) $u:[s,T] \to X$ *is continuous,*

(ii) $u:[s,T] \to X$ *is of class* $\mathcal{C}^1$,

(iii) $u(t) \in D$ *for each* $t \in (s,T]$,

(iv) $\frac{du}{dt}(t) + A(t)u(t) = f(t)$ *for each* $t \in (s,T]$.

2. Semigroups with singularity

We will use the following assumptions (see [6]):

(**Z**1). The domain D of $A(t)$ is independent on t, D is dense in X, and for $x \in D$ the function $[0,T] \ni t \to A(t)x \in X$ is of class $\mathcal{C}^1$.

(**Z**2). For all $t \in [0,T]$, $\lambda \in \sum_{b_0}^{\omega} = \{\lambda \in \mathbb{C} : |\arg(\lambda - b_0)| < \omega\}$, where $b_0 < 0$, $\omega \in \left(\frac{\pi}{2}, \pi\right)$ there exist $R_\lambda(t) = -(\lambda + A(t))^{-1} \in B(X)$ and there exist constants $M \geq 1$, $\theta \in \left(\frac{2}{3}, 1\right)$, independent on t and λ, such that

$$\|R_\lambda(t)\| \leq \frac{M}{(1+|\lambda|)^\theta} \quad \text{for} \quad \lambda \in \sum_{b_0}^{\omega},\ t \in [0,T].$$

For a fixed $s \in [0,T]$ we may use results of [10] to get the following remark:

Remark 2. *For any fixed* $s \in [0,T]$ *operators*

$$S_s(t) := \frac{-1}{2\pi i} \int_{\Gamma b} e^{\lambda t} R_\lambda(s) d\lambda, \tag{2}$$

where $\Gamma_b = \left\{ re^{-i\omega} + b;\ 0 \le r < \infty \right\} \cup \left\{ re^{i\omega} + b;\ 0 \le r < \infty \right\}$, $b \in (b_0, +\infty)$, *form an analytic semigroup with singularity generated by* $(-A(s))$ (see [9]). *Moreover*

$$\frac{d^n}{dt^n} S_s(t) = (-1)^n A^n(s) S_s(t) \qquad \text{for} \qquad s \in [0,T],\ t > 0,$$

and for any fixed $s \in [0,T]$ *the estimate*

$$\left\| A^n(s) S_s(t) \right\| \le M_n t^{\theta-(n+1)} \qquad \text{for} \qquad s \in [0,T],\ t > 0, \tag{3}$$

holds, where M_n *depends only on* θ *and M* (*by assumption* (**Z**2)).

Definition 3. *Let us define* (see [9])

$$A^{-\beta}(s) := \frac{-1}{2\pi i} \int_{\Gamma b} (-\mu)^{-\beta} R_\mu(s) d\mu,$$

for $\beta \in (1-\theta, 1)$, $s \in [0,T]$, $b \in (b_0, 0)$, *where for the function* $(-\mu)^{-\beta}$ *we mean a branch whose arguments lie between* $-\beta\pi$ *and* $\beta\pi$ *and which is analytic in the region obtained by omitting the positive real axis* (for details see [9]).

Remark 4. *Under assumptions* (**Z**1), (**Z**2) *the linear operator* $A^{-\beta}(s)$ *is bounded and injective. Thus* $A^{\beta}(s) := (A^{-\beta}(s))^{-1}$ *is well defined and for* $\beta \in (1-\theta, \theta)$ *we have* $D(A^{\beta}(s)) \supset D$ (for details see [9]).

We can write the main theorem of [6] in the following way:

Theorem 5. *Assumptions* (**Z**1), (**Z**2) *guarantire that the problem*

$$\begin{cases} \dfrac{du}{dt}(t) + A(t)u(t) = f(t), & t \in (s,T],\ s \in [0,T], \\ u(s) = x, & x \in \widehat{D^{\beta}} = \cup_{r \in [0,T]} D(A^{\beta}(r)),\ \beta \in (2(1-\theta), \theta), \end{cases}$$

has the unique two-parameter family of bounded operators $\left\{ U(t,s) \right\}_{(t,s) \in \Delta_T}$, *where* $\Delta_T = \left\{ (s,t) : 0 \le s < t \le T \right\}$ (*called fundamental solution*) *such that*:

1° $U(t,s) : X \to D$ *is bounded and*

$$\left\| U(t,s) \right\| \le C \left| t-s \right|^{\theta-1} \qquad \textit{for} \qquad s < t,$$

2° *For any fixed* $\beta \in (2(1-\theta), \theta)$

$$\lim_{s \to t} U(t,s)x = x \qquad \textit{for} \qquad x \in \bigcup_{r \in [0,T]} D(A^{\beta}(r)),$$

3° *The function* $(s,T] \ni t \to U(t,s)x$ *is of class* C^1 *for* $s \in [0,T]$, $x \in X$ *and*

$$\frac{\partial U}{\partial t}(t,s)x + A(t)U(t,s)x = 0 \qquad \textit{for} \qquad x \in X,\ 0 \le s < t \le T.$$

Moreover, $\left\| \frac{\partial U}{\partial t}(t,s) \right\| \le C \left| t-s \right|^{\theta-2}$ *and*

$$\left\| A(t)U(t,s)A^{-1}(s) \right\| \le C \left| t-s \right|^{\theta-1} \quad \text{for} \quad 0 \le s < t \le T.$$

4° *The function* $[0,t) \ni s \to U(t,s)x \in X$ *is differentiable for* $x \in D,\ t \in (0,T]$ *and*

$$\frac{\partial U}{\partial s}(t,s)x = U(t,s)A(s)x.$$

Remark 6. *In* [6] *it is proved that*

$$U(t,s) = S_s(t-s) + W(t,s) = S_s(t-s) + \int_s^t S_r(t-r)P(r,s)dr,$$

where $\{S_s(t)\}_{t>0}$ *is semigroup generated by* $(-A(s))$*, and operator* $P(r,s)$ *is linear and bounded on X.*

Lemma 7. (See [6]) *It the function f is continuous then the function*

$$G(t) = \int_s^t W(t,r)f(r)dr$$

is of class C^1 *and*

$$\frac{\partial}{\partial t}G(t) = \frac{\partial}{\partial t}\int_s^t W(t,r)f(r)dr = \int_s^t \int_r^t (A(t)S_t(t-p)P(t,r)$$
$$- A(p)S_p(t-p)P(p,r)dp\, f(r)dr + \int_s^t S_t(t-r)P(t,r)f(r)dr.$$

It is easy to proof the following technical lemma:

Lemma 8. *For each* $d \in (-1,\infty)$*, there is* $C > 0$ *such that for each* $a \in (0,\infty)$

$$\int_0^\infty t^d e^{-at} dt = Ca^{-d-1}. \tag{4}$$

3. Besov spaces

For definition of spaces of Besov functions and their properties we refer to [5] and [4]. We recall some of them. We need the following definitions for characterization of Besov functions.

Definition 9. (See [4, 5, 10]) *Let* $I = (a,b)$ *where* $-\infty < a < b < \infty$*. We define* $K_0(I)$ *as the set of functions* $\varphi : \mathbb{R}^2 \to \mathbb{R}$ *of class* C^∞*, such that* 1° *and* 2° *holds:*

1° *For all compact* $K \subset \mathbb{R}$ *there exists a compact set* $K_1 \subset \mathbb{R}$ *such that, supp* $\varphi(t,\cdot) \subset K_1$ *for* $t \in K$.

2° *For all compact* $K \subset \mathbb{R}$ *there exists compact set* $K_2 \subset I$ *such that supp* $\varphi(t,(t-\cdot)/\tau) \subset K_2$ *for* $t \in K$ *and* $\tau \in (0,1]$.

Definition 10. (See [5]) *Let I be as in the precedente definition*

$$K_m(I) := \left\{ \frac{\partial^m \varphi}{\partial s^m}(t,s) : \varphi \in K_0(I) \right\}.$$

Let $I=(a,b)$ be an interval and let the function φ_0 be of class C^∞ with support in I such that $\int \varphi_0(t)dt = 1$. Let us define for each integraf m functions

$$e_m(t,s) = \sum_{j=0}^{m-1} \frac{\partial^j}{\partial s^j}\left(\frac{1}{j!} s^j \varphi_0(t-s) \right),$$

$$e_m^*(t,s) = 2e_m(t,s) - \int e_m(t,r)e_m(t-cr,s-r)dr.$$

Theorem 11. (See [5] Theorem 1) *A distribution f is in* $B^\sigma_{p,q}(I)$, *if and only if exists* $m>\sigma$ *such that following conditions holds*

$$\left\langle \varphi\left(t, \frac{t-s}{c}\right), f(s) \right\rangle_s \in L^p(I,X,dt) \qquad \text{for all} \qquad \varphi \in K_0(I),$$

$$\tau^{-\sigma} \left\langle \tau^{-1}\varphi\left(t, \frac{t-s}{\tau}\right), f(s) \right\rangle_s \in L^q\left((0,c), L^p(I,X,dt), \frac{d\tau}{\tau} \right)$$

for all $\varphi \in K_m(I)$.

Theorem 12. (See [5] Theorem 2) *Let* $f \in B^\sigma_{\infty,1}(I,X)$, *where* $I=(a,b)$, *and let* $h,l \in \{0,1,2,3,\dots\}$, $-h<\sigma<l$, $m=h+l$. *Under above assumption the sequence* $\{f_n\}_{n=1}^\infty$ *defined by* $f_n(t) = \int_a^b n e_m^*(t,n(t-r)) f(r) dr$ *converges to f in* $B^\sigma_{\infty,1}(I,X)$. *If the function* $f:[a,b] \to X$ *is continuous then* $\{f_n\}_{n=1}^\infty$ *are in* $C^\infty([a,b]) \cap B^\sigma_{\infty,1}(I,X)$. *Moreover,* $\{f_n\}$ *converges to f in* $L^1(I,X)$. *By definition of norm in* $B^\sigma_{\infty,1}(I,X)$ *for* $\sigma>0$, *the sequence* f_n *converges to f in* $L^\infty((a,b);X)$, *while* $n \to \infty$.

4. The linear case

We consider the following linear Cauchy problem

$$\begin{cases} \dfrac{du}{dt}(t) + A(t)u(t) = f(t), & t \in (s,T],\ s \in [0,T] \\ u(s) = x, & x \in \widehat{D^\beta}, \end{cases} \tag{5}$$

where $f:[0,T]\to X$ is Besov function from $B^{1-\theta}_{\infty,1}$. We investigate the existence and uniqueness of the classical solution. We shall prove that function *u* given by

$$u(t)=U(t,s)x+\int_s^t U(t,r)f(r)dr$$

is the unique classical solution of the initial value problem.

We shall show that $u(\cdot)$ is differentiable. We note that $U(\cdot,s)x$ is the solution of initial value problem with *f* = 0. By the form of $U(t,s)x$ and by differentiability of $G(\cdot)$, given in Lemma 7, it is sufficient to show the differentiability of $F(\cdot)$, given by $F(t)=\int_s^t S_r(t-r)f(r)dr$. By the inclusion in Corollary 2.1 from [10] it is sufficient consider function $F\in B^1_{\infty,1}$. The following lemmas will show that *u* satisfies the equation. We shall prove this by aproximating the function *u* by solution of equations with the right-hand side function of class C^1. We can do that by Theorem 12.

Lemma 13. *Assume Z_1–Z_2 and let $q,z\in[0,T]$. Then there exists C independent of z and q, such that*

$$\left\|\frac{\partial^k}{\partial q^k}\frac{\partial}{\partial z}S_z(q)\right\|\le Cq^{2\theta-k-2} \tag{6}$$

Proof. By (2) and (**Z**2), we have

$$\left\|\frac{\partial^k}{\partial q^k}\frac{\partial}{\partial z}S_z(q)\right\|\le C\left\|\int_{\Gamma a}\frac{\partial^k}{\partial q^k}\frac{\partial}{\partial z}e^{\lambda q}R_\lambda(z)d\lambda\right\|\le Cq^{2\theta-k-2}(1+q^{1-\theta})\le$$
$$\le Cq^{2\theta-k-2}(1+T^{1-\theta})\le Cq^{2\theta-k-2}. \qquad \square$$

Lemma 14. *Let $\varphi\in K_0(s+\varepsilon,T)$, $\varepsilon>0$, $q_0\in(0,T-s)$, $q\in(0,T]$, $p\in(s+\varepsilon,T)$. Moreover, let function $f:[0,T]\to X$ be continuous and*

$$M=\sup\left\{\left|\varphi\left(p,\frac{p-q-r}{\tau}\right)\right|;\ p,q,r\in(0,T),\ \tau\in(o,1]\right\}.$$

Then

$$\left\|\int_s^{T-q_0}\int_z^{T-q_0}\tau^k\frac{\partial^{k+1}S_z}{\partial q^k\partial z}(q)\varphi\left(p,\frac{p-q_0-r}{\tau}\right)f(r)dr\,dz\right\|\le$$
$$\le C\tau^k q^{\theta-1-k}\left\|\int_s^{T-q_0}\varphi\left(p,\frac{p-q_0-r}{\tau}\right)f(r)dr\right\|+CM\tau^{k+2}q^{2\theta-2-k}\|f\|_{L^\infty}.$$

Proof. We shall show that $M<\infty$. For fixed $p\in(0,T)$ and $\tau(0,1]$ $\operatorname{supp}\ \varphi\left(p,\frac{p-\cdot}{\tau}\right)\subset[0,T]$.

$$\sup\left\{\left|\varphi\left(p,\frac{p-q_0-r}{\tau}\right)\right|;\ p,q_0,r\in(0,T),\ \tau\in(0,1]\right\}\le$$

$$\le\sup\left\{\left|\varphi\left(p,z-q_0-r\right)\right|;\ z,q_0,r\in(0,\tau T),\ p\in(0,T),\tau\in(0,1]\right\}<\infty$$

$$\varphi\in C^{\infty}\quad\text{hence}\quad\sup\left\{\varphi(t,s):(t,s)\in[0,T]\times[-2T,T]\right\}\le\infty.$$

Let $\sup\varphi\left(p,\frac{p-q_0-\cdot}{\tau}\right)\subset[a,b]\subset[q_0-K\tau,q_0+K\tau]$ i $[a,b]\subset(s+\varepsilon,T)$, where K is independent of τ. We have

$$\left\|\int_s^{T-q_0}\tau^k\frac{\partial^k}{\partial q^k}\frac{\partial S_z}{\partial z}(q)\int_z^{T-q_0}\varphi\left(p,\frac{p-q_0-r}{\tau}\right)f(r)dr\,dz\right\|\le$$

$$\le\left\|\int_s^{a}\tau^k\frac{\partial^k}{\partial q^k}\frac{\partial S_z}{\partial z}(q)\int_z^{T-q_0}\varphi\left(p,\frac{p-q_0-r}{\tau}\right)f(r)dr\,dz\right\|+$$

$$+\left\|\int_a^{b}\tau^k\frac{\partial^k}{\partial q^k}\frac{\partial S_z}{\partial z}(q)\int_z^{T-q_0}\varphi\left(p,\frac{p-q_0-r}{\tau}\right)f(r)dr\,dz\right\|+$$

$$+\left\|\int_b^{T-q_0}\tau^k\frac{\partial^k}{\partial q^k}\frac{\partial S_z}{\partial z}(q)\int_z^{T-q_0}\varphi\left(p,\frac{p-q_0-r}{\tau}\right)f(r)dr\,dz\right\|$$

Estimatig each term we prove Lemma 14. □

Lemma 15. (See [10]) *Let* $I=(s,T)$, $I_\varepsilon=(s+\varepsilon,T)$, $q',q\in(0,T]$, $\varphi\in K_0(I_\varepsilon)\cap K_0(I)$. *If* $f\in L_1(I,X)$ *then*

$$\left\|\int_s^{T-q}\tau^{-1}\varphi\left(t-q',\frac{t-q-r}{\tau}\right)f(r)dr\right\|_{L^\infty(I,X,dt)}\le C\tau^{-1}\|f\|_{L_1(I,X)}.$$

Lemma 16. (See [10]) *Let* $I=(s,T)$, $I_\varepsilon=(s+\varepsilon,T)$, $0\le q\le\varepsilon$. *If* $f\in L^1(I,X)$ *and* $\varphi\in K_0(I_\varepsilon)\cap K_0(I)$ *then*

$$\left\|\int_s^{T-q}\tau^{-1}\varphi\left(t,\frac{t-q-r}{\tau}\right)f(r)dr\right\|_{L^\infty(I_\varepsilon,X,dt)}\le$$

$$\le\sum_{j=0}^{2}\frac{q^j}{j!}\left\|\int_s^{T-q}\tau^{-1}\varphi_{j,0}\left(t,\frac{t-r}{\tau}\right)f(r)dr\right\|_{L^\infty(I,X,dt)}+Cq^3\tau^{-1}\|f\|_{L_1(I,X)},$$

where $\phi_{i,j}(t,s)=\frac{\partial^{i+j}}{\partial^i t\partial^j s}\phi(t,s)$.

Proof. By Taylor formula

$$\varphi(t,q')=\sum_{j=0}^{2}\frac{q^j}{j!}\varphi_{j,0}(t-q,q')+\frac{q^3}{2}\int_0^1\eta^2\varphi_{3,0}(t-\eta q,q')d\eta, \tag{7}$$

for $t\in I_\varepsilon$ support $\varphi\left(t,\frac{t-q-\cdot}{\tau}\right)\subset(s+\varepsilon-q,T-q)\subset(s,T-q)$ so integrals over $(s,T-q)$ and over (s,T) are equal. By Lemma 6 follows (7). □

Lemma 17. *Let function* $f:[s,T]\to X$ *be continuous function in Besov space* $B^{1-\theta}_{\infty,1}(s;T)$ *and* $\varphi\in K_0(I_\varepsilon)\cap K_0(I)$. *Then for every* $c\in(0,1]$ *we have*

$$\int_s^T\varphi\left(p,\frac{p-t}{c}\right)F(t)dt\in L^\infty(I_\varepsilon;X;dp), \tag{8}$$

where $F:[s,T]\to X$ *is given by*

$$F(t)=\int_s^t S_r(t-r)f(r)dr. \tag{9}$$

Proof. By (9) we have

$$\int_s^T\varphi\left(p,\frac{p-t}{c}\right)F(t)dt=\int_s^T\int_r^T\varphi\left(p,\frac{p-t}{c}\right)S_r(t-r)f(r)dt\,dr=$$
$$=\int_s^T\int_0^{T-r}\varphi\left(p,\frac{p-q-r}{c}\right)S_r(q)f(r)dq\,dr.$$

By assumption $S_{(\cdot)}(q)x$ is of class C^1. So

$$\int_s^T\varphi\left(p,\frac{p-t}{c}\right)F(t)dt=\int_s^T\int_0^{T-r}\int_s^r\frac{\partial}{\partial z}S_z(q)\varphi\left(p,\frac{p-q-r}{c}\right)f(r)dz\,dq\,dr+$$
$$+\int_s^T\int_0^{T-r}S_s(q)\varphi\left(p,\frac{p-q-r}{c}\right)f(r)dq\,dr.$$

The second term is in $L^\infty(I_\varepsilon;X;dp)$ (proof as in [10] because $A(s)$ is constant). By Remark 6 and Lemma 14 for the first one we have

$$\left\|\int_0^{T-s}\int_s^{T-q}\int_z^{T-q}\frac{\partial}{\partial z}S_z(q)\varphi\left(p,\frac{p-q-r}{c}\right)f(r)dr\,dz\,dq\right\|\le$$
$$\le C(T-s)^\theta\|f\|_{L^1}+Cc^2(T-s)^{2\theta-1}\|f\|_{L^\infty}.$$

□

Lemma 18. *Let continuous function* $f:[s,T]\to X$ *be in* $B^{1-\theta}_{\infty,1}(s;T)$, *and function* $F:[s,T]\to X$ *be defined by* (9). *Then there exists* $c\in(0;1]$ *such that for every* $\varphi\in K_5(I_\varepsilon)\cap K_5(I)$ *the following condition holds*

$$\tau^{-3}\int_s^T \varphi\left(p,\frac{p-t}{\tau}\right)F(t)dt \in L^{-1}((0,c);\ L^\infty(I_\varepsilon;X;dt);d\tau).$$

Proof. By definition of F and using the fact that $S_{(\cdot)}(q)x$ is of class C^1 we have

$$J=\tau^{-3}\int_s^T \varphi\left(p,\frac{p-t}{\tau}\right)F(t)dt=\tau^{-3}\int_s^T\int_0^{T-r}\varphi\left(p,\frac{p-q-r}{\tau}\right)S_r(q)f(r)dq\,dr=$$
$$=\tau^{-3}\int_s^T\int_0^{T-r}\int_s^r\frac{\partial}{\partial z}S_z(q)\varphi\left(p,\frac{p-q-r}{\tau}\right)f(r)dz\,dq\,dr+$$
$$+\tau^{-3}\int_s^T\int_0^{T-r}S_s(q)\varphi\left(p,\frac{p-q-r}{\tau}\right)f(r)dz\,dq\,dr.$$

The second term is in $L^\infty(I_\varepsilon;X;dp)$ (proof as in [10] $A(s)$ is constant). We consider the first term. Changing the order of integration we have

$$\tau^{-3}\int_s^T\int_0^{T-r}\int_s^r\frac{\partial}{\partial z}S_z(q)\varphi\left(p,\frac{p-q-r}{\tau}\right)f(r)dz\,dq\,dr=$$
$$=\tau^{-3}\left(\int_0^\tau+\int_\tau^{T-s}\right)\int_s^{T-q}\int_z^{T-q}\frac{\partial}{\partial z}S_z(q)\varphi\left(p,\frac{p-q-r}{\tau}\right)f(r)dr\,dz\,dq=J_1+J_2.$$

We estimate J_1 by Lemma 14 and Lemma 16

$$\|J_1\|\le C\sum_{j=0}^{2}\tau^j\left\|\int_s^T\tau^{-3+\theta}\varphi_{j,0}\left(p,\frac{p-r}{\tau}\right)f(r)dr\right\|_{L^\infty(I,X)}dq+$$
$$+C\tau^\theta\|f\|_{L_1(I,X)}+C\tau^{2\theta-2}\|f\|_{L^\infty}.$$

By assumption $f\in B^{1-\theta}_{\infty,1}$ we have that $J_1\in L^1((0,c);\ L^\infty(I_\varepsilon;X;dt);\ d\tau)$. We estimate J_2 in similiar way using Lemma 14 and Lemma 16.

Hence $J\in L^1((0,c);\ L^\infty(I_\varepsilon;X;dt);\ d\tau)$ as sum of two elements of this space. □

Remark 19. *By proofs of above lemmas the following inequality holds*

$$\|F\|_{B^1_{\infty,1}}\le C\|f\|_{B^{1-\theta}_{\infty,1}}+C\|f\|_{L^1}+C\|f\|_{L^\infty}. \tag{10}$$

Theorem 20. *Assume* Z_1-Z_4 *and let continuous function* $f:[0,T]\to X$ *be in* $B^{1-\theta}_{1,\infty}(0,T)$. *Then problem* (5) *has the unique solution, given by*

$$u(t)=U(t,s)x+\int_s^t U(t,r)f(r)dr. \tag{11}$$

Proof. $U(t,s)x$ is the solution of problem (5) with $f\equiv 0$. So, the first term in (11) is of class C^1. We consider the second term

$$\int_s^t U(t,r)f(r)dr = \int_s^t S_s(t-r)f(r)dr + \int_s^t W(t,r)f(r)dr.$$

The function f is continous. So Remark 6 gives that $\int_s^t W(\cdot,r)f(r)dr$ is of class C^1. By Lemma 17 and Lemma 18, also, $F(t) = \int_s^t S_s(t-r)f(r)dr$ is of class C^1.

Theorem 12 implies the existence of sequence $\{f_n\}_{n=1}^{\infty}$, such that

$$f_n \in B_{\infty,1}^{1-\theta} \cap C^1([s,T];X)$$

and

$$f_n \to f \qquad in \qquad B_{\infty,1}^{1-\theta} \cap L^1((s,T);X) \cap L^{\infty}([s,T];X).$$

Let us denote

$$F_n(t) = \int_s^t S_s(t-r)f_n(r)dr,$$

$$u_n(t) = U(t,s)x + \int_s^t U(t,r)f_n(r)dr = U(t,s)x + F_n(t) + G_n(t),$$

where $G(t) = \int_s^t W(t,r)f(r)dr$. Then

$$u_n \in C^1((s,T];X),\ u_n(t) \in D \qquad for \qquad t \in (s,T]$$

$$\frac{du_n}{dt}(t) = -A(t)u_n(t) + f_n(t) \qquad for \qquad t \in (s,T]. \tag{12}$$

Substituting f by $f - f_n$ in inequality (10) we have

$$\| F - F_n \|_{B_{\infty,1}^1} \le C\| f - f_n \|_{B_{\infty,1}^{1-\theta}} + C\| f - f_n \|_{L^1} + C\| f - f_n \|_{L^{\infty}}.$$

So

$$\| F - F_n \|_{B_{\infty,1}^{1-\theta}} \to 0, \qquad while \qquad n \to \infty.$$

On the other hand, by (12), we have

$$\left\| A(\cdot)u_n(\cdot) + \frac{du}{dt}(\cdot) + f(\cdot) \right\|_X \le$$

$$\le \left\| -f_n(\cdot) - \frac{dF_n}{dt}(\cdot) + \frac{dF}{dt}(\cdot) + f(\cdot) + \frac{dG}{dt}(\cdot) - \frac{dG_n}{dt}(\cdot) \right\|_X \le$$

$$\le \| f(\cdot) - f_n(\cdot) \|_X + \left\| \frac{dF}{dt}(\cdot) - \frac{dF_n}{dt}(\cdot) \right\|_X + \left\| \frac{dG}{dt}(\cdot) - \frac{dG_n}{dt}(\cdot) \right\|_X .$$

By Theorem 3 in [5], Remark 19 and definition of Besov function

$$\| f_n(\cdot) - f(\cdot) \|_{B^0_{\infty,1}} \le C \| f_n(\cdot) - f(\cdot) \|_{B^{1-\theta}_{\infty,1}} \to 0, \quad when \quad n \to \infty$$

$$\left\| \frac{dF}{dt}(\cdot) - \frac{dF_n}{dt}(\cdot) \right\|_{B^0_{\infty,1}} \le \| F(\cdot) - F_n(\cdot) \|_{B^1_{\infty,1}} \to 0, \quad when \quad n \to \infty.$$

By Lebesgue convergence theorem and Remark 7

$$\left\| \frac{dG}{dt}(t) - \frac{dG_n}{dt}(t) \right\|_X \to 0, \quad when \quad n \to \infty.$$

By closedness of $A(t)$ for $t \in (s,T]$ the theorem is proved.

□

5. Example

We give example of continuous functions

$$f : [0,1] \times [0,1] \to (-\infty, \infty)$$

and $u : [0,1] \to [0,1],$ such that

I. For fixed r function f satisfies the Lipschitz condition with respect to the second variable $\exists L \ge 0 \forall r \in [0,1] \forall t_1, t_2 \in [0,1]$

$$| f(r,t_1) - f(r,t_2) | \le L | t_1 - t_2 |.$$

II. For fixed t function $f(\cdot,t)$ is in Besov space $B^\sigma_{\infty,1}$, where $\sigma \in (0,1)$, i.e. $\exists M \ge 0$

$$\int_0^1 h^{-1-\sigma} \sup_{t\in[0,T-h]} | f(t+h,r) - f(t,r) | dh \le M.$$

III. Function u is of class C^∞.

IV. Composition function $f(\cdot, u(\cdot)) : [0,1] \to (-\infty,\infty)$ is not in Besov space, that is

$$\int_0^1 h^{-1-\sigma} \sup_{t\in[0,T-h]} | f(t+h, u(t+h)) - f(t,t) | dh = \infty.$$

This shows that the theorem on existence of the solution, for semilinear case requires additional assumption.

6. Construction

Let us define

$$p_n(t) = \begin{cases} \dfrac{2^{-n\sigma}}{n\log^2 2} & \text{for} \quad t = 0, \\ \dfrac{2^{-n\sigma}}{n\log^2 2} - \dfrac{nt^\sigma}{\log^2 t} & \text{for} \quad |t| \in \left(0, \dfrac{1}{2^n}\right], \\ 0 & \text{for} \quad |t| \geq \dfrac{1}{2^n}, \end{cases}$$

$$q_n(t,r) = \begin{cases} \left(1 - \dfrac{n\log^2 2}{2^{-n\sigma}}|r|\right) p_n(t) & \text{for} \quad |r| \leq \dfrac{2^{-n\sigma}}{n\log^2 2}, \\ 0 & \text{for} \quad |r| > \dfrac{2^{-n\sigma}}{n\log^2 2}, \end{cases}$$

$$f(t,r) = \begin{cases} q_{n_m}\left(t - \dfrac{3}{4}2^{1-m}, r - \dfrac{3}{4}2^{1-m}\right) & \text{for} \quad t \in \left(\dfrac{1}{2^m}, \dfrac{1}{2^{m-1}}\right], \\ 0 & \text{for} \quad t = 0, \end{cases}$$

where $n_m = k + km, \ k \geq \max\left\{2, \dfrac{1}{\sigma}\right\}$.

For fixed r easy computations shows that

$$\| f \|_{B^\sigma_{\infty,1}} \int_0^1 h^{-1-\sigma} \sup_{t\in[0,1-h]} |f(t+h,r) - f(t,r)| \, dh \leq 6.$$

For fixed r semi-norm of function $f(\cdot,r)$ in Besov function $B^\sigma_{\infty,1}$ is no greather than 6. Function f satisfies the Lipschitz condition with respect to the second variable with constant 1.

Let $u(t) = t$. Then $u(\cdot)$ is of class C^∞ and

$$\int_0^1 h^{-1-\sigma} \sup_{t\in[0,T-h]} |f(t+h, t+h) - f(t,t)| \, dh = \infty.$$

This means that $f(\cdot, u(\cdot)) \notin B^\sigma_{\infty,1}$.

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