

**TECHNICAL
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**FUNDAMENTAL
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**WYDAWNICTWO
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MATH

KRZYSZTOF WESOŁOWSKI*

FIXED POINTS OF α -NONEXPANSIVE MAPPINGSPUNKTY STAŁE ODWZOROWAŃ α -NIEODDALAJĄCYCH

Abstract

This paper is connected with the theory of α -nonexpansive mappings, which were introduced by K. Goebel and M. A. J. Pineda in 2007. These mappings are a natural generalisation of nonexpansive mappings from the point of view of the fixed point theory. In particular, they proved that in Banach spaces all $\alpha = (\alpha_1, \dots, \alpha_n)$ -nonexpansive mappings with α_1 big enough,

namely $\alpha_1 \geq \frac{1}{2^{1-n}}$, have minimal displacement equal to zero. This paper introduces some new results connected with this problem.

Keywords: α -nonexpansive mappings, minimal displacement, fixed point

Streszczenie

Niniejszy artykuł jest związany z odwzorowaniami α -nieoddalającymi, które zostały wprowadzone przez K. Goebła i M. A. J. Pinedę w 2007 r. Odwzorowania te są naturalnym uogólnieniem odwzorowań nieoddalających z punktu widzenia teorii punktu stałego. Wyżej wspomniani autorzy wykazali, że w przestrzeniach Banacha odwzorowania $\alpha = (\alpha_1, \dots, \alpha_n)$ -nieoddalające,

mające odpowiednio duże α_1 , a dokładniej $\alpha_1 \geq \frac{1}{2^{1-n}}$, posiadają minimalne przesunięcie równe zeru. W artykule przedstawiono pewne nowe wyniki z związane z tym problemem.

Słowa kluczowe: odwzorowania α -nieoddalające, minimalne przesunięcie, punkt stały

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1. Introduction and preliminaries

Let (X, d) be a metric space, and let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multi-index satisfying $\alpha_1 > 0$, $\alpha_n > 0$, $\alpha_i \geq 0$, $i = 2, \dots, n-1$ and $\sum_{i=1}^n \alpha_i = 1$. In [2], the following notions were introduced:

The mapping $T: X \rightarrow X$ is said to be α -Lipschitzian with constant $k \geq 0$, if

$$\sum_{i=1}^n \alpha_i d(T^i x, T^i y) \leq k d(x, y) \text{ for all } x, y \in X.$$

The mapping $T: X \rightarrow X$ is said to be α -nonexpansive (α -contraction), if T is α -Lipschitzian with constant $k = 1$ ($k < 1$ resp.).

Denote the Lipschitz constant with $k(T)$ and the α -Lipschitz constant of T with $k(\alpha, T)$.

Define also $d(T) := \inf\{d(x, Tx), x \in X\}$, which we will call the minimal displacement of T . Sometimes it is also called the approximate fixed point of T .

These notions are natural generalisations of Lipschitzian mappings, nonexpansive mappings and contractions from the point of view of the fixed point theory. For more information concerning α -nonexpansive mappings and other Lipschitzian mappings connected with the fixed point theory, we refer to [4].

In [2], the authors proved the following:

Theorem 1.1. (see also [4], chapter 3) *Let X be a Banach space, let C be a nonempty, closed, convex and bounded subset of X . Let $T: C \rightarrow C$ be an $\alpha = (\alpha_1, \dots, \alpha_n)$ -*

-nonexpansive mapping where $\alpha_1 \geq 2^{\frac{1}{1-n}}$. Then $d(T) = 0$.

Notice that the problem of determining the set of multi-indices α for which each α -nonexpansive mapping T has $d(T) = 0$ is still open.

The aim of this paper is to prove two results (Theorem 2.1, Theorem 2.2) which give a partial answer to the above open problem (see [4]).

Before proceeding further, let us recall the generalised Banach contraction principle (abbreviated to GBCP), which is formulated as follows:

Theorem 1.2. ([1], [5]) *In complete metric space X if for some $N \geq 1$ and $0 < M < 1$ the mapping $T: X \rightarrow X$ satisfies $\min\{d(T^j x, T^j y), 1 \leq j \leq N\} \leq M d(x, y)$ for any $x, y \in X$, then T has the unique fixed point.*

In author's PhD thesis [6] the more general version of the above theorem was presented. Let us recall it without proof.

Theorem 1.3. *Let (X, d) be a complete metric space, $N \geq 1$. Assume that $\phi: [0, \infty) \rightarrow [0, 1]$ is a continuous, non-increasing function satisfying $\phi(t) = 1$ if, and only if, $t = 0$. Let $T: X \rightarrow X$ be such that $\min\{d(T^j x, T^j y), 1 \leq j \leq N\} \leq \phi(d(x, y)) \cdot d(x, y)$ for all $x, y \in X$. Then T has the unique fixed point.*

2. Main results

Firstly, let us note a simple fact, there exist some α -Lipschitzian mappings which are not α -nonexpansive; however, their minimal displacement is equal to zero; moreover, they may have the unique fixed point.

This is illustrated by:

Example 2.1. Let $T : l_\infty \cap \{x \in l_\infty : x_i \geq 0, i \in \mathbb{N}\} \rightarrow l_\infty \cap \{x \in l_\infty : x_i \geq 0, i \in \mathbb{N}\}$

be defined in the following way: $T : x = (x_1, x_2, \dots) \rightarrow Tx := \left(1, \frac{2x_3}{1+x_3}, \frac{\frac{1}{2}x_2}{1+x_2}, \frac{2x_5}{1+x_5}, \frac{\frac{1}{2}x_4}{1+x_4}, \dots \right)$. Then T is not α -nonexpansive for any α ; however, for properly chosen

$\alpha = (\alpha_1, \alpha_2)$ the mapping T is α -Lipschitzian with constant k arbitrarily close to 1. Moreover, T has the unique fixed point.

Obviously, the mapping T has the unique fixed point $(1, 0, 0, \dots)$.

Also, we have $\|Tx - Ty\| \leq 2\|x - y\|$ and $\|T^i x - T^i y\| \leq \|x - y\|, i \geq 2$ for any $x, y \in l_\infty \cap \{x \in l_\infty : x_i \geq 0, i \in \mathbb{N}\}$. On the other hand, taking $x^n = \left(0, 0, \frac{1}{n}, 0, 0, \dots\right)$ and

$y^n = (0, 0, \dots)$ we have $\frac{\|Tx^n - Ty^n\|}{\|x^n - y^n\|} = \frac{2 \cdot \frac{1}{n}}{1 + \frac{1}{n}} n = \frac{2}{n+1} \rightarrow 2, n \rightarrow \infty$; therefore, $k(T) = 2$.

Similarly, $k(T^2) = 1$ and $k(T^i) \geq 1, i \geq 3$.

In l_∞ , it is not possible to choose α such that $\alpha_1 > 0$ and T is α -nonexpansive; however, $\frac{1}{n}\|Tx - Ty\| + \frac{n-1}{n}\|T^2x - T^2y\| \leq \frac{n+1}{n}\|x - y\|$; therefore, assuming n to be big enough, the mapping T is $\alpha = \left(\frac{1}{n}, \frac{n-1}{n}\right)$ -Lipschitzian with constant $k(\alpha, T)$ arbitrarily close to 1.

It is worth mentioning that the existence and uniqueness of the fixed point of T also follows from Theorem 1.3. Indeed, we have $T^2x = \left(1, \frac{x_2}{1 + \frac{3}{2}x_2}, \frac{x_3}{1 + 3x_3}, \dots\right)$

$$\left. \frac{x_4}{1+\frac{3}{2}x_4}, \frac{x_5}{1+3x_5}, \dots \right) \quad \text{and} \quad \left| \frac{x_i}{1+\frac{3}{2}x_i} - \frac{y_i}{1+\frac{3}{2}y_i} \right| \leq \left| \frac{x_i - y_i}{1+\frac{3}{2}(x_i + y_i) + \frac{9}{4}x_i y_i} \right| \leq \frac{|x_i - y_i|}{1+|x_i + y_i|} \leq$$

$$\leq \frac{|x_i - y_i|}{1+|x_i + y_i|} \leq \frac{\|x - y\|}{1+\|x - y\|}, \quad i \in \mathbb{N}. \quad \text{The latter inequality follows from the fact that } t \rightarrow \frac{t}{1+t}$$

is an increasing function on $[0, \infty)$. Similarly, $\left| \frac{x_i}{1+3x_i} - \frac{y_i}{1+3y_i} \right| \leq \frac{\|x - y\|}{1+\|x - y\|}$; therefore,

$$\|T^2x - T^2y\| \leq \frac{1}{1+\|x - y\|} \|x - y\|, \quad \text{so } T \text{ satisfies the assumptions of Theorem 1.3 with}$$

$$\phi(t) := \frac{1}{1+t}.$$

Now, let us exchange the condition $\alpha_1 \geq 2^{\frac{1}{1-n}}$ with the other regularity condition of a mapping T .

Theorem 2.1. *Let X be a Banach space, let $0 \in C \subset X$ be nonempty, closed, convex and bounded. Let $T : C \rightarrow C$ be a $\alpha = (\alpha_1, \dots, \alpha_n)$ -nonexpansive mapping such that $\|T^i(\mu x) - T^i(\mu y)\| \leq \|T^i(\lambda x) - T^i(\lambda y)\|$ for any $x, y \in C$, $0 \leq \mu \leq \lambda$, $i \in \{1, \dots, n\}$.*

Then $d(T) = 0$.

Proof. Fix $k \geq 1$. Define $S_k := \left(1 - \frac{1}{k}\right)T$. Obviously, $S_k x = \frac{1}{k} \cdot 0 + \left(1 - \frac{1}{k}\right)Tx \in C$. Then

$$\|S_k x - S_k y\| = \left(1 - \frac{1}{k}\right) \|Tx - Ty\|.$$

Next, we have:

$$S_k^2 x = S_k \left(\left(1 - \frac{1}{k}\right)Tx \right) = \left(1 - \frac{1}{k}\right)T \left(\left(1 - \frac{1}{k}\right)Tx \right);$$

therefore, by assumptions:

$$\begin{aligned} \|S_k^2 x - S_k^2 y\| &= \left\| \left(1 - \frac{1}{k}\right)T \left(\left(1 - \frac{1}{k}\right)Tx \right) - \left(1 - \frac{1}{k}\right)T \left(\left(1 - \frac{1}{k}\right)Ty \right) \right\| \\ &= \left(1 - \frac{1}{k}\right) \left\| T \left(\left(1 - \frac{1}{k}\right)Tx \right) - T \left(\left(1 - \frac{1}{k}\right)Ty \right) \right\| \\ &\leq \left(1 - \frac{1}{k}\right) \|T^2x - T^2y\|. \end{aligned}$$

Similarly, for $i \geq 3$ we have:

$$S_k^i x = T_k^{i-1} \left(\left(1 - \frac{1}{k} \right) T x \right) = \left(1 - \frac{1}{k} \right) T \left(\left(1 - \frac{1}{k} \right) T \left(\dots \left(\left(1 - \frac{1}{k} \right) T x \right) \dots \right) \right);$$

therefore:

$$\begin{aligned} \|S_k^i x - S_k^i y\| &= \left\| \left(1 - \frac{1}{k} \right) T \left(\dots \left(\left(1 - \frac{1}{k} \right) T x \right) \dots \right) - \left(1 - \frac{1}{k} \right) T \left(\dots \left(\left(1 - \frac{1}{k} \right) T y \right) \dots \right) \right\| \\ &= \left(1 - \frac{1}{k} \right) \left\| T \left(\left(1 - \frac{1}{k} \right) T \left(\dots \left(\left(1 - \frac{1}{k} \right) T x \right) \dots \right) \right) - T \left(\left(1 - \frac{1}{k} \right) T \left(\dots \left(\left(1 - \frac{1}{k} \right) T y \right) \dots \right) \right) \right\| \\ &\leq \left(1 - \frac{1}{k} \right) \left\| T \left(T \left(\dots \left(\left(1 - \frac{1}{k} \right) T x \right) \dots \right) \right) - T \left(T \left(\dots \left(\left(1 - \frac{1}{k} \right) T y \right) \dots \right) \right) \right\| \\ &\leq \left(1 - \frac{1}{k} \right) \|T^i x - T^i y\|. \end{aligned}$$

By assumptions, $\sum_{i=1}^n \alpha_i \|T^i x - T^i y\| \leq \|x - y\|$ for some $\alpha = (\alpha_1, \dots, \alpha_n)$ satisfying

$0 \leq \alpha_i \leq 1$ and $\sum_{i=1}^n \alpha_i = 1$. Therefore:

$$\min \left\{ \|S_k^j x - S_k^j y\|, 1 \leq j \leq n \right\} \leq \sum_{i=1}^n \alpha_i \|S_k^i x - S_k^i y\| \leq \left(1 - \frac{1}{k} \right) \|x - y\|,$$

By Theorem 1.2, S_k has the unique fixed point. Denote this fixed point by x_k . We get:

$$\|x_k - T x_k\| = \|S_k x_k - T x_k\| = \left\| \left(1 - \frac{1}{k} \right) T x_k - T x_k \right\| = \frac{1}{k} \|T x_k\| \rightarrow 0, \quad k \rightarrow \infty,$$

this completes the proof. □

For $k \geq 3$, there exists a mapping T which does not satisfy the assumptions of Theorem 1.1; however, for $k \geq 2$, it satisfies the assumptions of Theorem 2.1. This will be illustrated by the following example:

Example 2.2. Fix $k \geq 2$. Let $\tau: [-1, 1] \rightarrow [-1, 1]$ be a non-decreasing function, having the Lipschitz constant $k(\tau) = k$, concave on $[-1, 0]$, convex on $[0, 1]$ and such that

$$\tau(0) = 0. \text{ Now define } T: B_{l_1} \ni x = (x_1, x_2, \dots) \rightarrow T x := (\tau(x_2), \frac{k}{k^2 - 1} x_3, x_4, x_5, \dots) \in B_{l_1}.$$

We will show that the assumptions of Theorem 1.1 are not satisfied for any multi-index α of length n . Notice, that:

$$\begin{aligned}
\|Tx - Ty\| &= |\tau(x_2) - \tau(y_2)| + \frac{k}{k^2 - 1} |x_3 - y_3| + \sum_{i=4}^{\infty} |x_i - y_i| \\
&\leq k |x_2 - y_2| + \frac{k}{k^2 - 1} |x_3 - y_3| + \sum_{i=4}^{\infty} |x_i - y_i|, \\
\|T^2x - T^2y\| &= \left| \tau\left(\frac{k}{k^2 - 1}x_3\right) - \tau\left(\frac{k}{k^2 - 1}y_3\right) \right| + \frac{k}{k^2 - 1} |x_4 - y_4| + \sum_{i=5}^{\infty} |x_i - y_i| \\
&\leq \frac{k}{k^2 - 1} |x_3 - y_3| + \frac{k}{k^2 - 1} |x_4 - y_4| + \sum_{i=5}^{\infty} |x_i - y_i|.
\end{aligned}$$

Therefore, $k(T) = k$ and $k(T^i) = \frac{k^2}{k^2 - 1} > 1$, $i \geq 2$.

It is easy to see that for $k \geq 3$, the assumptions of Theorem 1.1 are not satisfied for any multi-index α of length n . Indeed, we would need to have $\alpha_1 \geq 2^{\frac{1}{1-n}} \geq 2^{\frac{1}{1-3}} = \frac{\sqrt{2}}{2} > \frac{1}{2}$ and thus for any such $\alpha = (\alpha_1, \dots, \alpha_n)$, the mapping T would not be α -nonexpansive.

We will now show that T satisfies the assumptions of Theorem 2.1; therefore, $d(T) = 0$.

It is enough to take $n = 2$. It is easy to check that $\frac{1}{k} \|Tx - Ty\| + \frac{k-1}{k} \|T^2x - T^2y\| \leq \|x - y\|$; therefore, T is $\left(\frac{1}{k}, \frac{k-1}{k}\right)$ -nonexpansive. We only have to show that

$$\|T(\mu x) - T(\mu y)\| \leq \|T(\lambda x) - T(\lambda y)\| \text{ for any } x, y \in B_{l_i}, 0 \leq \mu \leq \lambda.$$

If $|\tau(\mu x_2) - \tau(\mu y_2)| \leq |\tau(\lambda x_2) - \tau(\lambda y_2)|$, then obviously,

$$\begin{aligned}
\|T(\mu x) - T(\mu y)\| &= |\tau(\mu x_2) - \tau(\mu y_2)| + \left| \frac{k}{k^2 - 1} \mu x_3 - \frac{k}{k^2 - 1} \mu y_3 \right| + \sum_{i=4}^{\infty} |\mu x_i - \mu y_i| \\
&\leq |\tau(\lambda x_2) - \tau(\lambda y_2)| + \left| \frac{k}{k^2 - 1} \lambda x_3 - \frac{k}{k^2 - 1} \lambda y_3 \right| + \sum_{i=4}^{\infty} |\lambda x_i - \lambda y_i| \\
&= \|T(\lambda x) - T(\lambda y)\|
\end{aligned}$$

It is enough to prove that $|\tau(\mu v) - \tau(\mu w)| \leq |\tau(\lambda v) - \tau(\lambda w)|$ for $v \neq w$ and $0 < \mu < \lambda$. Firstly, assume that $v, w > 0$. Without the loss of generality, we can assume that $w < v$. Therefore, $0 < \mu w < \mu v, \lambda w < \lambda v$.

Assume that $0 < \mu w < \mu v \leq \lambda w < \lambda v$. Choose $a \in (\lambda w, \lambda v]$ such that $a - \lambda w = \mu v - \mu w$.

Of course, such an a exists since $\lambda(v - w) \geq \mu(v - w)$. Therefore, $\mu v = \frac{a - \mu w}{a - \mu w} \mu w + \frac{\mu v - \mu w}{a - \mu w} a$ and $\lambda w = \frac{a - \lambda w}{a - \mu w} \mu w + \frac{\lambda w - \mu w}{a - \mu w} a$. Due to the convexity of τ on $[0, 1]$

$$\begin{aligned} \tau(\mu v) &= \tau\left(\frac{a - \mu w}{a - \mu w} \mu w + \frac{\mu v - \mu w}{a - \mu w} a\right) \\ &\leq \frac{a - \mu w}{a - \mu w} \tau(\mu w) + \frac{\mu v - \mu w}{a - \mu w} \tau(a) \\ \tau(\lambda w) &= \tau\left(\frac{a - \lambda w}{a - \mu w} \mu w + \frac{\lambda w - \mu w}{a - \mu w} a\right) \\ &\leq \frac{a - \lambda w}{a - \mu w} \tau(\mu w) + \frac{\lambda w - \mu w}{a - \mu w} \tau(a) \\ &= \frac{\mu v - \mu w}{a - \mu w} \tau(\mu w) + \frac{a - \mu w}{a - \mu w} \tau(a) \end{aligned}$$

Adding the above estimates side-by-side and taking into consideration the fact that τ is non-decreasing, we get:

$$\begin{aligned} \tau(\mu v) + \tau(\lambda w) &\leq \left(\frac{a - \mu w}{a - \mu w} + \frac{\mu v - \mu w}{a - \mu w}\right) \tau(\mu w) + \left(\frac{\mu v - \mu w}{a - \mu w} + \frac{a - \mu w}{a - \mu w}\right) \tau(a) \\ &= \tau(\mu w) + \tau(a) \leq \tau(\mu w) + \tau(\lambda v), \end{aligned}$$

this implies that $|\tau(\mu v) - \tau(\mu w)| \leq |\tau(\lambda v) - \tau(\lambda w)|$.

On the other hand, if $0 < \mu w < \lambda w \leq \mu v < \lambda v$, then let us choose $a \in (\mu v, \lambda v]$ such that $a - \mu v = \lambda w - \mu w$. Of course, such an a exists since $(\lambda - \mu)v \geq (\lambda - \mu)w$. Then

$\lambda w = \frac{a - \lambda w}{a - \mu w} \mu w + \frac{\lambda w - \mu w}{a - \mu w} a$ and $\mu v = \frac{a - \mu v}{a - \mu w} \mu w + \frac{\mu v - \mu w}{a - \mu w} a$. Due to the convexity of τ on $[0, 1]$, we have:

$$\begin{aligned} \tau(\lambda w) &= \tau\left(\frac{a - \lambda w}{a - \mu w} \mu w + \frac{\lambda w - \mu w}{a - \mu w} a\right) \\ &\leq \frac{a - \lambda w}{a - \mu w} \tau(\mu w) + \frac{\lambda w - \mu w}{a - \mu w} \tau(a) \end{aligned}$$

$$\begin{aligned}
\tau(\mu\nu) &= \tau\left(\frac{a-\mu\nu}{a-\mu w}\mu w + \frac{\mu\nu-\mu w}{a-\mu w}a\right) \\
&\leq \frac{a-\mu\nu}{a-\mu w}\tau(\mu w) + \frac{\mu\nu-\mu w}{a-\mu w}\tau(a) \\
&= \frac{\lambda w-\mu w}{a-\mu w}\tau(\mu y) + \frac{a-\lambda w}{a-\mu w}\tau(a)
\end{aligned}$$

Again, adding the above estimations side-by-side, we get:

$$\begin{aligned}
\tau(\lambda w) + \tau(\mu\nu) &\leq \left(\frac{a-\lambda w}{a-\mu w} + \frac{\lambda w-\mu w}{a-\mu w}\right)\tau(\mu w) + \left(\frac{\lambda w-\mu w}{a-\mu w} + \frac{a-\lambda w}{a-\mu w}\right)\tau(a) \\
&= \tau(\mu w) + \tau(a) \leq \tau(\mu w) + \tau(\lambda\nu),
\end{aligned}$$

this leads to $|\tau(\mu\nu) - \tau(\mu w)| \leq |\tau(\lambda\nu) - \tau(\lambda w)|$.

Similarly, it is easy to check that the estimation $|\tau(\mu\nu) - \tau(\mu w)| \leq |\tau(\lambda\nu) - \tau(\lambda w)|$ remains true for $\nu, w < 0$ and for other cases. This shows that T satisfies the assumptions of Theorem 2.1.

A set satisfying $\lambda x_0 + (1-\lambda)y \in C$ for all $y \in C$, $\lambda \in [0,1]$ we call *star-like set* C with respect to x_0 .

Theorem 2.2. *Let X be a Banach space, $x_0 \in X$, $N \in \mathbb{N}$, let $C \subset X$ be a bounded, star-like set with respect to x_0 . Let $T : C \rightarrow C$ be such that*

1. $\min\{\|T^j x - T^j y\|, 1 \leq j \leq N\} \leq \|x - y\|$ for all $x, y \in C$,
2. there exists $0 \leq b_0 \leq 1$ such that for all $0 \leq b \leq b_0$, $1 \leq j \leq N-1$, $x, y \in C$

$$\|T(T_b^j x) - T(T_b^j y)\| \leq (1+b)\|T^{j+1}x - T^{j+1}y\|,$$

where $T_b x = (1-b)Tx + bx_0$. Then $d(T) = 0$.

Proof. Fix arbitrary $x, y \in C$ and take $j \in \{1, \dots, N\}$ such that $\|T^j x - T^j y\| \leq \|x - y\|$.

Let us note that:

$$\begin{aligned}
\|T_b^j x - T_b^j y\| &= \|T_b(T_b^{j-1}x) - T_b(T_b^{j-1}y)\| \\
&= \|(1-b)T(T_b^{j-1}x) + bx_0 - (1-b)T(T_b^{j-1}y) - bx_0\| \\
&= (1-b)\|T(T_b^{j-1}x) - T(T_b^{j-1}y)\| \\
&\leq (1-b)(1+b)\|T^j x - T^j y\| \\
&\leq (1-b^2)\|x - y\|
\end{aligned}$$

Therefore, for any $x, y \in C$ there exists $j \in \{1, \dots, N\}$ such that $\|T_b^j x - T_b^j y\| \leq (1-b^2)\|x - y\|$. Theorem 1.2 ensures, that T_b has the unique fixed point.

Now, fix an arbitrary $\varepsilon > 0$ and choose $0 \leq b \leq b_0$ such that $\|T_b z - Tz\| = \|(1-b)Tz + bx_0 - Tz\| = b\|x_0 - Tz\| \leq \varepsilon$ for any $z \in C$.

Let $z_b \in C$ be such that $T_b z_b = z_b$.

Therefore, $\|z_b - Tz_b\| \leq \|z_b - T_b z_b\| + \|T_b z_b - Tz_b\| \leq 0 + \varepsilon = \varepsilon$, this proves that $d(T) = 0$.

□

Let us illustrate the possible application of Theorem 2.2.

Example 2.3. Let T be the same as in Example 2.2. Then T satisfies Theorem 2.2 (we have already shown that T does not satisfy Theorem 1.1 for $k \geq 3$).

Indeed, let us calculate

$$T(T_b x) = \left(\tau \left((1-b) \frac{k}{k^2-1} x_3 \right), \frac{k}{k^2-1} (1-b)x_4, (1-b)x_5, (1-b)x_6, \dots \right)$$

and

$$T^2 x = \left(\tau \left(\frac{k}{k^2-1} x_3 \right), \frac{k}{k^2-1} x_4, x_5, x_6, \dots \right)$$

We have:

$$\begin{aligned} \|T(T_b x) - T(T_b y)\| &= \left| \tau \left((1-b) \frac{k}{k^2-1} x_3 \right) - \tau \left((1-b) \frac{k}{k^2-1} y_3 \right) \right| \\ &\quad + \left| \frac{k}{k^2-1} (1-b)x_4 - \frac{k}{k^2-1} (1-b)y_4 \right| + |(1-b)x_5 - (1-b)y_5| + \dots \\ &\leq \left| \tau \left(\frac{k}{k^2-1} x_3 \right) - \tau \left(\frac{k}{k^2-1} y_3 \right) \right| + \left| \frac{k}{k^2-1} x_4 - \frac{k}{k^2-1} y_4 \right| + |x_5 - y_5| + \dots \\ &= \|T^2 x - T^2 y\| \leq (1+b) \|T^2 x - T^2 y\|. \end{aligned}$$

We have already taken into account the fact, that $|\tau(\mu s) - \tau(\mu t)| \leq |\tau(\lambda s) - \tau(\lambda t)|$ for any $0 \leq \mu \leq \lambda$, $s, t \in [-1, 1]$. We proved this fact in Example 2.2.

The estimate $\min \left\{ \|Tx - Ty\|, \|T^2 x - T^2 y\| \right\} \leq \frac{1}{k} \|Tx - Ty\| + \frac{k-1}{k} \|T^2 x - T^2 y\| \leq \|x - y\|$ shows that T satisfies Theorem 2.2.

References

- [1] Arvanitakis A.D., *A proof of the generalized Banach contraction conjecture*, Proc. of the Amer. Math. Soc. **131**, 2003, 12:3647-3656, MR 1998170.
- [2] Goebel K., Japon Pineda M.A., *On a type of generalized nonexpansiveness*, Proceedings of the 8th International Conference of Fixed Point Theory and its Applications, 2007, 71-82.
- [3] Goebel K., Sims B., *Mean Lipschitzian Mappings*, Contemporary Mathematics, **513**, 2010, 157-167.
- [4] Piasecki L., *Classification of Lipschitz mappings*, Pure and applied mathematics (307), CRC Press, Boca Raton, FL, 2014.
- [5] Merryfeld J., Stein J.D. Jr., *A generalization of the Banach contraction principle*, J. Math. Anal. Appl. **273**, 2002, 112-120, MR 1933019 (2003g:54100).
- [6] *Punkty stałe odwzorowań w przestrzeniach metrycznych i SF-przestrzeniach*, Thesis, Jagiellonian University, Kraków 2015.

KRZYSZTOF WESOŁOWSKI*

SOME REMARKS ON THE GENERALIZED BANACH CONTRACTION PRINCIPLE

KILKA UWAG O UOGÓLNIONYM TWIERDZENIU BANACHA O PUNKCIE STAŁYM

Abstract

This paper presents some results concerning the Generalized Banach Contraction Principle: *In a complete metric space X if for some $N \geq 1$ and $0 < M < 1$ the mapping $T : X \rightarrow X$ satisfies $\min\{d(T^j x, T^j y), 1 \leq j \leq N\} \leq Md(x, y)$ for any $x, y \in X$, then T has a unique fixed point.* In some special cases, the above constant M can be replaced by a continuous, non-increasing function $0 \leq \phi(d(x, y)) \leq 1$ such that $\phi(t) = 1$ if, and only if, $t = 0$.

Keywords: generalized Banach contraction principle, gbcp, fixed point, metric fixed point theory, syndetic set

Streszczenie

W artykule przedstawiono pewne wyniki związane z Uogólnionym Twierdzeniem Banacha o Punkcie Stałym: *W przestrzeni metrycznej zupełnej X jeśli dla pewnych $N \geq 1$ i $0 < M < 1$ odwzorowanie $T : X \rightarrow X$ spełnia warunek $\min\{d(T^j x, T^j y), 1 \leq j \leq N\} \leq Md(x, y)$ dla dowolnych $x, y \in X$, to T ma dokładnie jeden punkt stały.* W pewnych szczególnych przypadkach zastąpiono stałą M ciągłą, nierosnącą funkcją $0 \leq \phi(d(x, y)) \leq 1$ dla której $\phi(t) = 1$ wtedy i tylko wtedy, gdy $t = 0$.

Słowa kluczowe: uogólnione twierdzenie Banacha o punkcie stałym, gbcp, punkt stały, metryczna teoria punktu stałego, zbiór syndetyczny

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1. Introduction

This paper is related to the Generalized Banach Contraction Principle (we will refer to it as GBCP for short) formulated as follows:

Theorem 1.1. [1] *In a complete metric space X if for some $N \geq 1$ and $0 < M < 1$ the mapping $T : X \rightarrow X$ satisfies $\min\{d(T^j x, T^j y), 1 \leq j \leq N\} \leq Md(x, y)$ for any $x, y \in X$, then T has a unique fixed point.*

Firstly, let us have a quick look at a history of this theorem.

For $N = 1$ the above theorem is obviously the celebrated classical Banach contraction principle, see [B].

In [2], it was shown that if X is compact and if for $x, y \in X$ and $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $d(T^n x, T^n y) \leq \varepsilon$, then T has a unique fixed point.

In [7], it was shown that GBCP is true for $N = 2$ and for $N = 3$, if T is continuous.

In [6], it was shown that it is true for any $N \geq 1$, if T is uniformly continuous.

In [11], it was shown that it is true for any $N \geq 1$, if T is strongly continuous.

In [8], it was shown that it is true for $N = 3$ and for any $N \geq 1$, if T is continuous.

In [9] and [1] the above theorem was finally proved.

Before we proceed, let us recall some definitions:

Finite set $A = \{n_1 < n_2 < \dots < n_k\} \subset \mathbb{N}$ is said to be S -syndetic with constant $S \in \mathbb{N}$, if $n_{i+1} - n_i \leq S$ for $1 \leq i \leq k-1$.

Infinite set $A \subset \mathbb{N}$ is said to be S -syndetic with constant $S \in \mathbb{N}$, if for any $l \in \mathbb{N}$, there is $\{i \in \mathbb{N} : l+1 \leq i \leq l+S\} := [l+1, l+S] \cap A \neq \emptyset$.

Infinite set $A \subset \mathbb{N}$ is said to be piecewise S -syndetic with constant $S \in \mathbb{N}$, if for any $N \in \mathbb{N}$, there exists $B \subset A$ such that $\#B \geq N$ and B is S -syndetic.

We call set A syndetic (or piecewise syndetic), if it is S -syndetic (piecewise S -syndetic) for some $S \in \mathbb{N}$.

Both proofs of Theorem 1.1 presented in [9], [1] are not elementary. In [1], the proof uses, among other tools, the strong result of H. Fürstenberg stated that if $\mathbb{N} \supset B = \bigcup_{i=1}^N B_i$ is piecewise syndetic, then B_i is piecewise syndetic for some $i \in \{1, \dots, N\}$. In [9], the proof uses Ramsey's theorem.

Another approach to the metric fixed point theory, presented in ([10]), resulted in the following:

Theorem 1.2. [10] *Let (X, d) be a complete metric space. Let $\phi : [0, \infty) \rightarrow [0, 1]$ be a non-increasing function such that $\phi(t) = 1$ if, and only if, $t = 0$. Assume that $T : X \rightarrow X$ is a contractive mapping such that:*

$$1. \quad d(x_0, Tx_0) \leq \frac{d(x, x_0) - d(Tx, Tx_0)}{2}, \quad x \notin M$$

$$2. \quad d(Tx, Ty) \leq \phi(d(x, y))d(x, y), \quad x, y \in M$$

for some $M \subset X$, $x_0 \in M$. Then, T has a unique fixed point.

This instantly led to the following result regarding contractive mappings in complete metric spaces:

Corollary 1.3. *For ϕ as in Theorem 1.2, if $d(Tx, Ty) \leq \phi(d(x, y))d(x, y)$, $x, y \in X$, then T has the unique fixed point.*

2. Main results

In this paper, we try to mix two approaches presented in Theorem 1.1 and Corollary 1.3, in order to achieve, under some additional assumptions, the following:

Theorem 2.1. *Let (X, d) be a complete metric space, $N \geq 1$. Let $\phi: [0, \infty) \rightarrow [0, 1]$ be a continuous, non-increasing function satisfying $\phi(t) = 1$ if, and only if, $t = 0$. Assume that $T: X \rightarrow X$ satisfies $\min\{d(T^j x, T^j y), 1 \leq j \leq N\} \leq \phi(d(x, y)) \cdot d(x, y)$, $x, y \in X$. Then T has a unique fixed point.*

The above theorem, in this general form, was proved in the author's PhD thesis [12]. The proof is mainly based on the above mentioned Fürstenberg theorem. In this paper, we will show some special but important cases in which it is not necessary to apply this strong tool, which significantly simplifies the proof.

Firstly, we assume that T is a continuous mapping having the N -syndetic Cauchy orbit.

Theorem 2.2. *Let (X, d) be a complete metric space, $N \geq 1$. Let $T: X \rightarrow X$ be a continuous mapping such that:*

1. *for all $y \neq z \in X$, there exists $n \in \mathbb{N}_1$ such that $d(T^n y, T^n z) < d(y, z)$*
2. *there exists $x \in X$, $N \in \mathbb{N}_1$ and $\{n_j\}_{j=1}^\infty \subset \mathbb{N}$ -increasing, N -syndetic such that*

$\{T^{n_j} x\}_{j=1}^\infty$ is a Cauchy sequence.

T then has exactly one fixed point.

Proof. Let $x_0 := \lim_{n \rightarrow \infty} T^{n_j} x$. Since $1 \leq n_{j+1} - n_j \leq N$, there exists $b \in \{1, \dots, N\}$, $\{n_{j_l}\} \subset \{n_j\}$ such that $n_{j_l+1} - n_{j_l} = b$. On the other hand, $T^{j_l+1} x = T^b(T^{n_{j_l}} x)$. By definition of x_0 , there is $T^{j_l+1} x \rightarrow x_0$, $l \rightarrow \infty$, and due to the continuity of T , we have $T^b(T^{n_{j_l}} x) \rightarrow T^b(x_0)$, $l \rightarrow \infty$. Therefore, T^b has a fixed point x_0 .

Let $c := \min\{b \in \mathbb{N}_1 : T^b \text{ has a fixed point}\}$. Let us assume that $c < 1$.

Let $r := \min\{d(T^j x_0, T^k x_0), 0 \leq j < k \leq c-1\}$. Of course, $r > 0$ – the opposite case would contradict with the choice of c . Since the minimum was taken over a finite set, there exists $j_0, k_0 \in \{0, \dots, c-1\}$ such that $j_0 < k_0$ and $d(T^{j_0} x_0, T^{k_0} x_0) = r$.

By assumption, we can choose $n \in \mathbb{N}$ such that $d(T^{n+j_0} x_0, T^{n+k_0} x_0) < d(T^{j_0} x_0, T^{k_0} x_0)$. But $n + j_0 = p \cdot c + \tilde{j}_0$. On the other hand, $n + k_0 = q \cdot c + \tilde{k}_0$ for some $\tilde{j}_0, \tilde{k}_0 \in \{0, \dots, c-1\}$. Of course, $\tilde{j}_0 \neq \tilde{k}_0$. Moreover, $T^{q \cdot c} x_0 = T^c(\dots(T^c x_0)\dots) = x_0 = T^c(\dots(T^c x_0)\dots) = T^{p \cdot c} x_0$. Therefore, $d(T^{\tilde{j}_0} x_0, T^{\tilde{k}_0} x_0) = d(T^{p \cdot c + \tilde{j}_0} x_0, T^{q \cdot c + \tilde{k}_0} x_0) = d(T^{n+j_0} x_0, T^{n+k_0} x_0) < d(T^{j_0} x_0, T^{k_0} x_0)$, which leads to a contradiction with the choice of j_0, k_0 .

We will now show that there is exactly one fixed point. Let us assume on the contrary, that there are $x_0 \neq y_0$ such that $Tx_0 = x_0$, $Ty_0 = y_0$. By our assumption, there exists $n \in \mathbb{N}$ such that $d(x_0, y_0) > d(T^n x_0, T^n y_0) = d(x_0, y_0)$, which leads to a contradiction. \square

If there exists $x_0 \in X$ such that $d(T^n x_0, T^{n+1} x_0) \rightarrow 0$ with $n \rightarrow \infty$, we can directly point out the Cauchy orbit. This fact is a subject of the following:

Theorem 2.3. *Let (X, d) be a complete metric space. Assume that $\phi: \mathbb{R}_+ \rightarrow [0, 1]$ is a continuous, non-increasing function such that $\phi(t) = 1$ if, and only if, $t = 0$. Let $T: X \rightarrow X$ be a mapping satisfying the following conditions:*

1. *there exists $N \in \mathbb{N}_1$ such that $\min\{d(T^j x, T^j y), 1 \leq j \leq N\} \leq \phi(d(x, y))d(x, y)$ for all $x, y \in X$,*
2. *$d(T^{n+1} x, T^n x) \rightarrow 0$, $n \rightarrow \infty$ for some $x \in X$.*

Then $\{T^n x\}_{n=1}^\infty$ is a Cauchy sequence.

Proof. Let $x_n := T^n x$. Assume for indirect proof, that $\{T^n x\}_{n=1}^\infty = \{x_n\}_{n=1}^\infty$ is not a Cauchy sequence. Therefore, there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, there exists $k_n, l_n > n$ such that $d(x_{k_n}, x_{l_n}) \geq \varepsilon$. For $n \in \mathbb{N}$ let $k_n > n$ be the smallest possible such that there exist $l_n > k_n$ satisfying $d(x_{k_n}, x_{l_n}) \geq \varepsilon$. For $k_n > n$, let l_n be the smallest possible such that

$$d(x_{k_n}, x_{l_n}) \geq \varepsilon. \text{ Fix } n_0 \in \mathbb{N} \text{ such that for all } n \geq n_0, \quad d(T^{n+1} x, T^n x) = d(x_{n+1}, x_n) \leq \frac{\varepsilon}{2}.$$

Let us notice, that for $n \geq n_0$ there is $l_n \geq k_n + 2$. Then, for fixed $l_n \geq k_n + 2$ we have $d(x_{k_n}, x_{l_n}) \leq d(x_{k_n}, x_{l_n-1}) + d(x_{l_n-1}, x_{l_n}) < \varepsilon + d(x_{l_n-1}, x_{l_n}) \rightarrow \varepsilon$ with $n \rightarrow \infty$. Therefore, $\limsup_{n \rightarrow \infty} d(x_{k_n}, x_{l_n}) \leq \varepsilon$. However, since $d(x_{k_n}, x_{l_n}) \geq \varepsilon$, we have $d(x_{k_n}, x_{l_n}) \rightarrow \varepsilon$, $n \rightarrow \infty$.

On the other hand, for all $n \in \mathbb{N}$ there exists $j_n \in \{1, \dots, N\}$ such that $d(x_{k_n+j_n}, x_{l_n+j_n}) \leq \phi(d(x_{k_n}, x_{l_n}))d(x_{k_n}, x_{l_n})$. Define $c_n^i := i \cdot \chi_{\{0, \dots, j_n-1\}}(i) \in \{0, \dots, N-1\}$, where χ is an indicator function. Let us estimate:

$$d(x_{k_n}, x_{l_n}) \leq d(x_{k_n}, x_{k_n+1}) + d(x_{k_n+1}, x_{l_n+1}) + d(x_{l_n+1}, x_{l_n}).$$

Repeating this procedure, we get:

$$\begin{aligned} d(x_{k_n}, x_{l_n}) &\leq \sum_{i=1}^{j_n-1} d(x_{k_n+i}, x_{k_n+i+1}) + d(x_{k_n+j_n}, x_{l_n+j_n}) + \sum_{i=1}^{j_n-1} d(x_{l_n+i+1}, x_{l_n+i}) \\ &= \sum_{i=1}^{N-1} d(x_{k_n+c_n^i}, x_{k_n+c_n^i+1}) + d(x_{k_n+j_n}, x_{l_n+j_n}) + \sum_{i=1}^{N-1} d(x_{l_n+c_n^i+1}, x_{l_n+c_n^i}) \\ &\leq \sum_{i=1}^{N-1} d(x_{k_n+c_n^i}, x_{k_n+c_n^i+1}) + \phi(d(x_{k_n}, x_{l_n}))d(x_{k_n}, x_{l_n}) + \sum_{i=1}^{N-1} d(x_{l_n+c_n^i+1}, x_{l_n+c_n^i}). \end{aligned}$$

By assumptions $d(x_{k_n+c_n^i}, x_{k_n+c_n^i+1}) \rightarrow 0$ and $d(x_{l_n+c_n^i+1}, x_{l_n+c_n^i}) \rightarrow 0$ with $n \rightarrow \infty$ for $i \in \{0, \dots, N-1\}$. Therefore, with $n \rightarrow \infty$ in the above estimates, due to the continuity of ϕ , we have $\varepsilon \leq 0 + \phi(\varepsilon) \cdot \varepsilon + 0$. This implies $\phi(\varepsilon) \geq 1$, which gives $\varepsilon = 0$ and we therefore get a contradiction. \square

Theorem 2.3 ensures the existence and uniqueness of a fixed point.

Theorem 2.4. *Mapping T , which fulfils the assumptions of Theorem 2.3, has exactly one fixed point.*

Proof. Using notations from Theorem 2.3, let $z_0 := \lim_{n \rightarrow \infty} T^n x_0$. Due to the properties of T for any $n \in \mathbb{N}$, there exists $j_n \in \{1, \dots, N\}$ such that $d(T^{n+j_n} x_0, T^{j_n} z_0) \leq \phi(d(T^n x_0, z_0))d(T^n x_0, z_0)$. Therefore, there exists $j \in \{1, \dots, N\}$ and $\{n_k\} \subset \mathbb{N}$ such that $d(T^{n_k+j} x_0, T^j z_0) \leq \phi(d(T^{n_k} x_0, z_0))d(T^{n_k} x_0, z_0)$ for $k \in \mathbb{N}$. Due to this fact, $T^{n_k+j} x_0 \rightarrow T^j z_0$ with $k \rightarrow \infty$. But on the other hand, $T^{n_k+j} x_0 \rightarrow z_0$, so $T^j z_0 = z_0$.

Let $Z := \{T^n x_0\}_{n=1}^\infty \cup \{z_0\}$. Let us notice, that $T^j(Z) \subset Z$.

We will show that z_0 is the only one fixed point of $T^j|_Z$. Indeed, if there is $z_0 \neq w_0 \in Z$ – another fixed point of T^j , then $w_0 = T^n x_0$ for some $n \in \mathbb{N}$. Then

$$w_0 = T^j w_0 = \dots = T^{k \cdot j} w_0 = T^{k \cdot j} (T^n x_0) = T^{k \cdot j + n} x_0 \rightarrow z_0, k \rightarrow \infty,$$

which leads to a contradiction.

We will show that z_0 is a fixed point of T . Let j be the smallest natural number such that $T^j z_0 = z_0$. Let $0 < \varepsilon := \min\{d(T^k z_0, T^l z_0), 0 \leq k < l \leq j-1\} = d(T^{k_0} z_0, T^{l_0} z_0)$ for

some $0 \leq k_0 < l_0 \leq j-1$. There exists $m \in \{1, \dots, N\}$ such that $\varepsilon \leq d(T^{k_0+m}z_0, T^{l_0+m}z_0) \leq \phi(d(T^{k_0}z_0, T^{l_0}z_0))d(T^{k_0}z_0, T^{l_0}z_0) = \phi(\varepsilon)\varepsilon$. Due to this fact, $\varepsilon = 0$ – this leads to a contradiction.

We will now show that z_0 is the only fixed point of T . Indeed, if there is $z_0 \neq w_0 \in X$ – another fixed point of T , then for any $n \in \mathbb{N}$ there is $T^n z_0 = z_0$ and $T^n w_0 = w_0$. For points z_0, w_0 , there exists $j \in \{1, \dots, N\}$ such that $d(z_0, w_0) = d(T^j z_0, T^j w_0) \leq \phi(d(z_0, w_0)) \cdot d(z_0, w_0)$. This inequality implies that either $d(z_0, w_0) = 0$ or $\phi(d(z_0, w_0)) < 1$. Due to the properties of ϕ , in both cases $d(z_0, w_0) = 0$ – this contradicts with $z_0 \neq w_0$.

□

The theorem below is useful for proving Theorem 2.1 for $N = 2$.

Theorem 2.5. *Let (X, d) be a complete metric space, $N \geq 1$ and let $\phi: [0, \infty) \rightarrow [0, 1]$ be a continuous, non-increasing function such that $\phi(t) = 1$ if, and only if, $t = 0$. Let $T: X \rightarrow X$ satisfy $\min\{d(T^j x, T^j y), 1 \leq j \leq N\} \leq \phi(d(x, y)) \cdot d(x, y)$ for any $x, y \in X$.*

Moreover, for fixed $x, y \in X$ let the sequence $\{j_n\}_{n=1}^\infty \subset \{1, \dots, N\}$ be chosen (by the above assumption) in such a way, that for $n \geq 2$ we have $d(T^{j_1+\dots+j_n}x, T^{j_1+\dots+j_n}y) \leq \phi(d(T^{j_1+\dots+j_{n-1}}x, T^{j_1+\dots+j_{n-1}}y)) \cdot d(T^{j_1+\dots+j_{n-1}}x, T^{j_1+\dots+j_{n-1}}y)$. Let $z_n := d(T^{j_1+\dots+j_n}x, T^{j_1+\dots+j_n}y)$. Then $z_n \rightarrow 0$ with $n \rightarrow \infty$.

Proof. Obviously $z_n \geq 0$. Since $z_n \leq \phi(z_{n-1})z_{n-1}$ and $\phi(t) \leq 1$, $\{z_n\}_{n=1}^\infty$, is non-increasing.

If there exists $n \in \mathbb{N}$ such that $z_n = z_{n+1}$, then $z_n = z_{n+1} \leq \phi(z_n)z_n$; therefore, either $z_n = 0$ or $\phi(z_n) \geq 1$. Due to the properties of ϕ , in both cases $z_n = 0$.

In the opposite case (when for every $n \in \mathbb{N}$ there is $z_n \neq z_{n+1}$) the sequence $\{z_n\}_{n=1}^\infty$ is decreasing and bounded from below; therefore, it has a limit. Let $g := \lim_{n \rightarrow \infty} z_n$. We will show that this limit is 0. Assume for the purpose of contradiction that $g > 0$. Since $\phi(g) < 1$ and $\phi(z_n) \rightarrow \phi(g)$, due to continuity of ϕ , for any $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ $\phi(z_n) \leq \phi(g) + \delta$. Choose δ such that $\phi(g) + \delta < 1$. Then $\phi(z_n) \leq \phi(g) + \delta < 1$ for $n \geq n_0$. On the other hand, $0 \leq z_n \leq \phi(z_{n-1})z_{n-1} \leq \phi(z_{n-1}) \dots \phi(z_{n_0})z_{n_0} \leq [\phi(g) + \delta]^{n-n_0} z_{n_0} \rightarrow 0$ with $n \rightarrow \infty$; therefore, $z_n \rightarrow 0$ with $n \rightarrow \infty$.

□

Now we will show proof of Theorem 2.1 for $N = 2$ without applying the Fürstenberg theorem.

Theorem 2.6. *Let (X, d) be a complete metric space. Let $\phi: \mathbb{R}_+ \rightarrow [0, 1]$ be a continuous, non-increasing function such that $\phi(t) = 1$ if, and only if, $t = 0$. Let $T: X \rightarrow X$ be a mapping such that $\min\{d(T^j x, T^j y), 1 \leq j \leq 2\} \leq \phi(d(x, y))d(x, y)$ for any $x, y \in X$.*

Then $d(T^{n+1}x_0, T^n x_0) \rightarrow 0$, $n \rightarrow \infty$ for any $x_0 \in X$ and T has exactly one fixed point.

Proof. Assume for indirect proof, that there exists $x_0 \in X$ such that $d(T^{n+1}x_0, T^n x_0) \not\rightarrow 0$; therefore, there are $x_0 \in X$, $\varepsilon > 0$, $\{k_n\}_{n=1}^\infty \subset \mathbb{N}$ such that $d(T^{k_n}x_0, T^{k_n+1}x_0) \geq \varepsilon$.

For $k = 1, 2$ let us create two sequences $\{n_m^k\}_{m=1}^\infty$ in the following way:

Let $n_0^k := 0$; Having already chosen n_0^k, \dots, n_m^k , define $z_m^k := d(T^{n_m^k}x_0, T^{n_m^k+k}x_0)$ and

$$n_{m+1}^k := \begin{cases} n_m^k + 1, & \text{gdy } d(T^{n_m^k+1}x_0, T^{n_m^k+1+k}x_0) \leq \phi(z_m^k) \cdot z_m^k \\ n_m^k + 2, & \text{gdy } d(T^{n_m^k+2}x_0, T^{n_m^k+2+k}x_0) \leq \phi(z_m^k) \cdot z_m^k \end{cases}$$

The sequences $\{n_m^k\}_{m=0}^\infty$, $k = 1, 2$ are increasing and 2-syndetic. Moreover, due to Theorem 2.5, we have $z_m^k \rightarrow 0$, $m \rightarrow \infty$, $k \in \{1, 2\}$.

Let us notice, that for all $m_0 \in \mathbb{N}$, there exists $m \geq m_0$ satisfying $k_m \notin \{n_p^1\}_{p=0}^\infty$. Otherwise, there exists $m_0 \in \mathbb{N}$ such that for any $m \geq m_0$, there is $k_m \in \{n_p^1\}_{p=0}^\infty$; therefore, $k_m = n_{p(m)}^1$. Taking, if needed, the subsequence of $\{k_m\}_{m=0}^\infty$ and renumbering it, without loss of generality, we can assume that $m \leq p(m)$. Then $d(T^{k_m}x_0, T^{k_m+1}x_0) = d(T^{n_{p(m)}^1}x_0, T^{n_{p(m)}^1+1}x_0) \rightarrow 0$, $m \rightarrow \infty$, which leads to contradiction.

Therefore, for all $m_0 \in \mathbb{N}$, there exists $m \geq m_0$ satisfying $k_m \notin \{n_p^1\}_{p=0}^\infty$. For such m , there exists $p(m) \in \mathbb{N}$ satisfying $n_{p(m)}^1 < k_m < n_{p(m)+1}^1$. Without the loss of generality, we can assume that $m \leq p(m)$. Obviously, $n_{p(m)}^1 + 1 = k_m = n_{p(m)+1}^1 - 1$.

Since $\{n_q^2\}_{q=0}^\infty$ is a 2-syndetic sequence, there exists $q(m) \in \mathbb{N}$ such that either $k_m = n_{q(m)}^2$ or $k_m - 1 = n_{q(m)}^2$, because $\{k_m - 1, k_m\} \cap \{n_q^2\}_{q=0}^\infty \neq \emptyset$. Without the loss of generality, we can assume that $m \leq q(m)$.

If $k_m - 1 = n_{q(m)}^2$, then $d(T^{k_m}x_0, T^{k_m+1}x_0) \leq d(T^{k_m}x_0, T^{k_m-1}x_0) + d(T^{k_m-1}x_0, T^{k_m+1}x_0) \leq d(T^{n_{p(m)}^1}x_0, T^{n_{p(m)}^1+1}x_0) + d(T^{n_{q(m)}^2}x_0, T^{n_{q(m)}^2+2}x_0) \rightarrow 0$, with $m \rightarrow \infty$ – this follows from Theorem 2.5, and leads to contradiction.

If $k_m = n_{q(m)}^2$, then $d(T^{k_m}x_0, T^{k_m+1}x_0) \leq d(T^{k_m}x_0, T^{k_m+2}x_0) + d(T^{k_m+2}x_0, T^{k_m+1}x_0) \leq d(T^{n_{q(m)}^2}x_0, T^{n_{q(m)}^2+2}x_0) + d(T^{n_{p(m)+1}^1}x_0, T^{n_{p(m)+1}^1+1}x_0) \rightarrow 0$, with $m \rightarrow \infty$ – this also leads to contradiction.

Therefore, $d(T^{n+1}x_0, T^n x_0) \rightarrow 0$ with $n \rightarrow \infty$. Theorem 2.4 completes the proof. \square

We will now show a proof of Theorem 2.1 for uniformly continuous mappings.

Theorem 2.7. *Let (X, d) be a metric space. Assume that $\phi: \mathbb{R}_+ \rightarrow [0, 1]$ is a continuous, non-increasing function such that $\phi(t) = 1$ if and only if $t = 0$. Let $T: X \rightarrow X$ be a uniformly continuous mapping such that*

$$\forall x, y \in X \quad \min\{d(T^j x, T^j y), 1 \leq j \leq N\} \leq \phi(d(x, y))d(x, y) \quad (1)$$

for some $N \geq 1$.

Moreover, if X is complete, then T has exactly one fixed point.

Proof. First we show, that $d(T^{n+1}x_0, T^n x_0) \rightarrow 0$, $n \rightarrow \infty$ for any $x_0 \in X$. Assume on the contrary, that $d(T^n x_0, T^{n+1}x_0) \not\rightarrow 0$ for some $x_0 \in X$; therefore, there exists $x_0 \in X$, $\varepsilon > 0$, $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$ such that:

$$d(T^{n_k}x_0, T^{n_k+1}x_0) > \varepsilon, \quad k \in \mathbb{N} \quad (2)$$

Due to the uniform continuity of T , for the ε chosen above, there exists $\delta > 0$ such that $d(Tx, Ty) < \varepsilon$ for any $x, y \in X$ satisfying $d(x, y) < \delta$.

Repeating this procedure N times, δ can be chosen in such way, that:

$$d(x, y) < \delta \text{ implies } d(T^j x, T^j y) < \varepsilon \text{ for any } 1 \leq j \leq N, \quad x, y \in X \quad (3)$$

Due to the assumption (1) for $x_0 \in X$ chosen in (2), there exists $j_1 \in \{1, \dots, N\}$ such that $d(T^{j_1}x_0, T^{j_1+1}x_0) \leq \phi(d(x_0, Tx_0))d(x_0, Tx_0)$. There then exists $j_2 \in \{1, \dots, N\}$ such that $d(T^{j_1+j_2}x_0, T^{j_1+j_2+1}x_0) \leq \phi(d(T^{j_1}x_0, T^{j_1+1}x_0))d(T^{j_1}x_0, T^{j_1+1}x_0)$.

Continuing the above procedure, we will get $\{j_1 + \dots + j_l\}_{l=1}^\infty$ – an increasing and N -syndetic sequence, which fulfils the assumptions of Theorem 2.5 for the pair (x_0, Tx_0) . Therefore, $d(T^{j_1+\dots+j_l}x_0, T^{j_1+\dots+j_l+1}x_0) \rightarrow 0$ with $l \rightarrow \infty$. Define $m_l := j_1 + \dots + j_l$, $l \in \mathbb{N}$.

Let $\widehat{l}_0 \in \mathbb{N}$ be such that $d(T^{m_l} x_0, T^{m_l+1} x_0) < \delta$ for $l \geq \widehat{l}_0$. Since the sequence $\{m_l\}_{l=\widehat{l}_0}^\infty$ is N -syndetic (because $\{m_l\}_{l=1}^\infty$ is an increasing); therefore, we can choose $l_0 \geq \widehat{l}_0$ such that $\{m_{l_0} + 1, \dots, m_{l_0} + N\} \cap \{n_k\}_{k=1}^\infty \neq \emptyset$.

Therefore, there exists $j_0 \in \{1, \dots, N\}$ and $k_0 \in \mathbb{N}$ such that:

$$m_{l_0} + j_0 = n_{k_0} \quad (4)$$

By (2), it follows that $d(T^{n_{k_0}} x_0, T^{n_{k_0}+1} x_0) > \varepsilon$, and also $d(T^{m_{l_0}} x_0, T^{m_{l_0}+1} x_0) < \delta$, it therefore (3) implies that $d(T^{m_{l_0}+j_0} x_0, T^{m_{l_0}+j_0+1} x_0) < \varepsilon$, which, in respect of (4), leads to contradiction.

□

Now we can make use of the above theorems in the following *example*:

Example 2.1. Let $T: l_\infty \cap \{x \in l_\infty : x_i \geq 0, i \in \mathbb{N}\} \rightarrow l_\infty \cap \{x \in l_\infty : x_i \geq 0, i \in \mathbb{N}\}$ be defined in the following way: $T: x = (x_1, x_2, \dots) \rightarrow Tx := \left(1, \frac{2x_3}{1+x_3}, \frac{\frac{1}{2}x_2}{1+x_2}, \frac{2x_5}{1+x_5}, \frac{\frac{1}{2}x_4}{1+x_4}, \dots\right)$.

Obviously, $\|Tx - Ty\|_\infty \leq 2\|x - y\|_\infty$. We also have $T^2x = \left(1, \frac{x_2}{1+\frac{3}{2}x_2}, \frac{x_3}{1+3x_3}, \dots\right)$.

$\frac{x_4}{1+\frac{3}{2}x_4}, \frac{x_5}{1+3x_5}, \dots$ Let us estimate:

$$\left| \frac{x_i}{1+\frac{3}{2}x_i} - \frac{y_i}{1+\frac{3}{2}y_i} \right| \leq \left| \frac{x_i - y_i}{1+\frac{3}{2}(x_i + y_i) + \frac{9}{4}x_i y_i} \right| \leq \frac{|x_i - y_i|}{1+|x_i + y_i|} \leq \frac{|x_i - y_i|}{1+|x_i - y_i|} \leq \frac{\|x - y\|_\infty}{1+\|x - y\|_\infty},$$

$i \in \mathbb{N}$.

The latter inequality follows from the fact that $t \rightarrow \frac{t}{1+t}$ is an increasing function

on $[0, \infty)$. Similarly, $\left| \frac{x_i}{1+3x_i} - \frac{y_i}{1+3y_i} \right| \leq \frac{\|x - y\|_\infty}{1+\|x - y\|_\infty}$, and therefore, $\|T^2x - T^2y\|_\infty \leq$

$\leq \frac{1}{1 + \|x - y\|_\infty} \|x - y\|_\infty$. The mapping T satisfies the assumptions of both Theorem 2.6

and Theorem 2.7 with $\phi(t) := \frac{1}{1+t}$, it therefore follows that T has a unique fixed point. It is of course $(1, 0, 0, \dots)$. However, the mapping T does not satisfy assumptions of Theorem 1.1.

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References

- [1] Arvanitakis A.D., *A proof of the generalized Banach contraction conjecture*, Proc. of the Amer. Math. Soc. **131**, 2003, 12:3647-3656, MR 1998170.
- [2] Bailey D.F., *Some theorems on contractive mappings*, J. London Math. Soc. **41**, 1966, 101-106.
- [3] Banach S., *Sur les operations dans les ensembles abstraits et leur application aux equations integrales*, Fund. Math. **3**, 1922, 133-181.
- [4] Furstenberg H., *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, 1981, MR0603625 (82j:28010).
- [5] Goebel K., *Concise course on fixed point theorems*, Yokohama Publishers, 2002.
- [6] Jachymski J.R., Stein J.D. Jr., *A minimum condition and some related fixed-point theorems*, J. Austral. Math. Soc. **66**, 1999, 224-243.
- [7] Jachymski J.R., Schroder B., Stein J.D. Jr., *A connection between fixed point theorems and tiling problems*, J. Combin. Theory **87**, 1999, 273-286.
- [8] Merryfeld J., Rothschild B., Stein J.D. Jr., *An application of Ramsey's theorem to the Banach contraction principle*, Proc. of the Amer. Math. Soc. **130**, 2001, 927-933, MR 2002h:54040.
- [9] Merryfeld J., Stein J.D. Jr., *A generalization of the Banach contraction principle*, J. Math. Anal. Appl. **273**, 2002, 112-120, MR 1933019 (2003g:54100).
- [10] Rakotch E., *A note on contractive mappings*, Proc. Amer. Math. Soc. **13**, 1962, 459-465.
- [11] Stein J.D. Jr., *A systematic generalization procedure for fixed-point theorems*, Rocky Mountain J. of Math., **30**, 2000, 735-754 (2), MR 2001i:54052.
- [12] *Punkty stałe odwzorowań w przestrzeniach metrycznych i SF-przestrzeniach*, Thesis, Jagiellonian University, Kraków 2015.

MAŁGORZATA ZAJĘCKA*

A GENERALISATION OF THE CHIRKA-SADULLAEV
THEOREM FOR COMPLEX MANIFOLDSUOGÓLNIENIE TWIERDZENIA CHIRKI-SADULLAEVA
DLA ROZMAITOŚCI ZESPOŁONYCH

Abstract

The aim of this paper is to generalize the Chirka-Sadullaev theorem, fundamental in the theory of extension of separately holomorphic functions with singularities, to the case of a σ -compact Josefson manifold.

Keywords: separately holomorphic function, extension with singularities, pluripolar set, complex manifold

Streszczenie

Celem pracy jest uogólnienie twierdzenia Chirki-Sadullaeva, podstawowego twierdzenia w teorii przedłużania funkcji oddzielnie holomorficzych z osobliwościami, na przypadek σ -zwartych rozmaitości Josefszona.

Słowa kluczowe: funkcja oddzielnie holomorficzna, przedłużanie z osobliwościami, zbiór pluripolarny, rozmaitość zespolona

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1. Introduction and prerequisites

In [1], the authors proved a deep result that is now considered fundamental in the theory of extensions of separately holomorphic functions with singularities.

Theorem 1.1. (Theorem 1 from [1]) Let f be a holomorphic function on the polydisk $U' \times U_n$ in $\mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$, and for each fixed a' in some nonpluripolar set $E \subset U'$, the function $f(a', z_n)$ can be continued holomorphically to the whole plane with the exception of some polar set of singularities $M(a') \subset \mathbb{C}$, then f can be continued holomorphically to $(U' \times \mathbb{C}) \setminus S$, where S is a closed pluripolar subset of $(U' \times \mathbb{C})$.

Theorem 1.1 remains true if U' is a domain of holomorphy in \mathbb{C}^{n-1} and U_n is a domain in \mathbb{C} . Moreover, a set S in the conclusion can be minimised in the sense that for any $a' \in E$, the fiber $S_{(a', \cdot)} := \{z_n \in \mathbb{C} : (a', z_n) \in S\}$ is contained in $M(a')$ and for any $z' \in U'$, the fiber $S_{(z', \cdot)}$ is polar (see, for instance, Theorem 9.2.24 in [5] for details).

Applications in mathematical tomography found by O. Öktem (see [7], [8]) revived an interest in the theory of extensions of separately holomorphic functions with singularities (see [2], [3], [4]) and possible generalisations of the theory to the general case of complex manifolds (see [6], [9]). However, one of the main problems in the development of the theory in the case of manifolds is the lack of analogs of many fundamental results. In this paper, Theorem 1.1 is generalised to the case of σ -compact Josefson manifolds.

Recall that a manifold is called σ -compact (or *countable at infinity*) if it is a union of countably many compact subsets. A complex manifold X is called a *Josefson manifold* if every locally pluripolar set in X is globally pluripolar.

2. Main Theorem

Theorem 2.1. Let D be a σ -compact connected Josefson complex manifold of dimension n and let $A \subset D$ not be pluripolar. Let Δ be a nonempty domain in \mathbb{C} . For any $a \in A$, let $M(a) \subset \mathbb{C}$ be a closed polar set such that $\Delta \cap M(a) = \emptyset$. Let $S \subset \mathcal{O}(D \times \Delta)$ be such that for any $f \in S$ and $a \in A$, there exists a function $\tilde{f}_a \in \mathcal{O}(\mathbb{C} \setminus M(a))$ such that $\tilde{f}_a(\cdot) = f(a, \cdot)$ on Δ .

Then there exists a closed pluripolar set $\widehat{M} \subset D \times \mathbb{C}$ such that:

- (a) $\widehat{M} \cap (D \times \Delta) = \emptyset$,
- (b) $\widehat{M}_{(a, \cdot)} \subset M(a)$, $a \in A$,
- (c) $\widehat{M}_{(z, \cdot)} \subset \mathbb{C}$ is polar, $z \in D$,
- (d) any $f \in S$ extends holomorphically to $\widehat{f} \in \mathcal{O}((D \times \mathbb{C}) \setminus \widehat{M})$.

Proof of Theorem 2.1.

Step 1.

Let U be a domain in D such that there exists a biholomorphism $\Phi : U \rightarrow \Omega_U$, where Ω_U is a domain in \mathbb{C}^n . Assume that $A_U := A \cap U$ is not pluripolar. Then $B := \Phi(A_U) \subset \Omega_U$ is not pluripolar. For any $b \in B$, define $M(b) := M(a)$, where $a = \Phi^{-1}(b) \in A_U$. Thus $M(b)$ is closed, polar and $M(b) \cap \Delta = \emptyset$. Define a family $S_U := \{f|_{U \times \Delta} : f \in S\}$. From the assumptions for any $f \in S_U$ and $a \in A_U$ there exists a function $\tilde{f}_{U,a} := \tilde{f}_a \in \mathcal{O}(\mathbb{C} \setminus M(a))$ such that $\tilde{f}_{U,a}(w) = f(a, w)$, $w \in \Delta$. Define a new family $\mathcal{F} := \{g(b, w) = f(\Phi^{-1}(b), w) : f \in S_U\}$. Then $\mathcal{F} \subset \mathcal{O}(\Omega_U \times \Delta)$ and for each $g \in \mathcal{F}$ and $b \in B$, a function $\tilde{g}_b(w) := \tilde{f}_{U, \Phi^{-1}(b)}(w)$ is, from its definition, holomorphic on $\mathbb{C} \setminus M(b)$ and $\tilde{g}_b(w) = g(\Phi^{-1}(b), w) = g(b, w)$ for $w \in \Delta$.

Hence, from Theorem 1.1, there exists a relatively closed pluripolar set $\tilde{M} \subset \Omega_U \times \mathbb{C}$ such that:

- $\tilde{M}_{(b, \cdot)} \subset M(b)$, $b \in B$,
- $\tilde{M}_{(b, \cdot)} \subset \mathbb{C}$ is polar, $b \in \Omega_U$,
- $\tilde{M} \cap (\Omega_U \times \Delta) = \emptyset$,
- for any $g \in \mathcal{F}$, there exists a function $\hat{g} \in \mathcal{O}((\Omega_U \times \mathbb{C}) \setminus \tilde{M})$ such that $g = \hat{g}$ on $\Omega_U \times \Delta$.

Define a set $\widehat{M} := \{(\Phi^{-1}(b), w) : (b, w) \in \tilde{M}\}$. Fix a $z \in U$ and let $b = \Phi(z)$. Then

$$\widehat{M}_{(z, \cdot)} = \{w \in \mathbb{C} : (z, w) \in \widehat{M}\} = \{w \in \mathbb{C} : (\Phi^{-1}(b), w) \in \tilde{M}\} = \{w \in \mathbb{C} : (b, w) \in \tilde{M}\} = \tilde{M}_{(b, \cdot)}$$

Hence, $\widehat{M}_{(a, \cdot)} \subset M(a)$ for each $a \in A_U$ and $\widehat{M}_{(z, \cdot)}$ is polar for $z \in U$. Now, assume that there exists a point $(a, w) \in \widehat{M}$ such that $(a, w) \in U \times \Delta$. Let $b = \Phi(a)$. Then $(b, w) \in \Omega_U \times \Delta$ and, since $(a, w) \in \{(\Phi^{-1}(b), w) : (b, w) \in \tilde{M}\}$, $(b, w) \in \tilde{M}$. Thus, $\tilde{M} \cap (\Omega_U \times \Delta) \neq \emptyset$ – a contradiction.

For fixed $f \in S_U$, define $\hat{f}(a, w) := \hat{g}(\Phi(a), w)$. Because $(a, w) \in \widehat{M}$ if, and only if, $(\Phi(a), w) \in \tilde{M}$, $\hat{f} \in \mathcal{O}((U \times \mathbb{C}) \setminus \widehat{M})$. Moreover, $\hat{f}(z, w) = \hat{g}(\Phi(z), w) = g(\Phi(z), w) = f(z, w)$ for $(z, w) \in U \times \Delta$, where last equality follows from the definition of the family \mathcal{F} . Thus, any function $f \in S$ has an extension

$$\hat{f}_U := \begin{cases} \hat{f} & \text{on } (U \times \mathbb{C}) \setminus \widehat{M}, \\ f & \text{on } D \times \Delta \end{cases},$$

which is well defined and holomorphic on $(D \times \Delta) \cup ((U \times \mathbb{C}) \setminus \widehat{M})$.

Step 2.

Let U denote a domain in D such that there exists a relatively closed pluripolar set $\widehat{M}_U \subset U \times \mathbb{C}$ with the following properties:

- (1) $(\widehat{M}_U)_{(a,\cdot)} \subset M(a)$, $a \in A \cap U$,
- (2) $(\widehat{M}_U)_{(z,\cdot)} \subset \mathbb{C}$ is polar, $z \in U$,
- (3) $\widehat{M}_U \cap (U \times \Delta) = \emptyset$,
- (4) for any $f \in S$, there exists an extension $\hat{f}_U \in \mathcal{O}((D \times \Delta) \cup ((U \times \mathbb{C}) \setminus \widehat{M}_U))$ such that $f = \hat{f}_U$ on $D \times \Delta$.

Let V be a domain in D , biholomorphic to a domain in \mathbb{C}^n , such that $V \cap U \neq \emptyset$. Define a family $S_V = \{f|_{V \times \Delta} : f \in S\}$. For any $a \in U \cap V$, define $M(a) := (\widehat{M}_U)_{(a,\cdot)}$. If the set $(V \cap A) \setminus U$ is not empty, for $a \in (V \cap A) \setminus U$ let $M(a)$ be as in the assumptions of Theorem 2.1. For any $f \in S_V$ for $a \in U \cap V$ define $\tilde{f}_{V,a}(\cdot) := \hat{f}_U(a, \cdot)$ and for $a \in (V \cap A) \setminus U$ (if not empty) let $\tilde{f}_{V,a} := \tilde{f}_a$ from the assumptions. From Step 1, with V playing the role of U and A_U replaced by $(V \cap U) \cup (A \cap V)$ ¹ there exists a relatively closed, pluripolar set $\widehat{M}_V \subset V \times \mathbb{C}$ such that:

- $(\widehat{M}_V)_{(a,\cdot)} \subset (\widehat{M}_U)_{(a,\cdot)} \subset M(a)$, $a \in A \cap V \cap U$,
- $(\widehat{M}_V)_{(a,\cdot)} \subset M(a)$, $a \in (A \cap V) \setminus U$,
- $(\widehat{M}_V)_{(z,\cdot)} \subset \mathbb{C}$ is polar, $z \in V$,
- $\widehat{M}_V \cap (V \times \Delta) = \emptyset$,
- for any $f \in S$, there exists $\hat{f}_V \in \mathcal{O}((D \times \Delta) \cup ((V \times \mathbb{C}) \setminus \widehat{M}_V))$ such that $f = \hat{f}_V$ on $D \times \Delta$.

Then a set $\widehat{M} := \widehat{M}_U \cup \widehat{M}_V \subset (U \cup V) \times \mathbb{C}$ is relatively closed and pluripolar, $\widehat{M} \cap (D \times \Delta) = \emptyset$, for each $a \in A$, the fiber $\widehat{M}_{(a,\cdot)} = (\widehat{M}_U)_{(a,\cdot)} \cup (\widehat{M}_V)_{(a,\cdot)} \subset M(a)$ and for any $z \in D$, $\widehat{M}_{(z,\cdot)}$ is polar. Fix $f \in S$. The function f has two extensions:

¹ $V \cap U$ is necessary, because $A \cap V$ alone could be pluripolar.

$\widehat{f}_U \in \mathcal{O}((D \times \Delta) \cup ((U \times \mathbb{C}) \setminus \widehat{M}_U))$ and $\widehat{f}_V \in \mathcal{O}((D \times \Delta) \cup ((V \times \mathbb{C}) \setminus \widehat{M}_V))$ such that on $D \times \Delta$ the equality $\widehat{f}_U = \widehat{f}_V = f$ holds. Define a function

$$\widehat{f} := \begin{cases} \widehat{f}_U & \text{on } (U \times \mathbb{C}) \setminus \widehat{M} \\ \widehat{f}_V & \text{on } (V \times \mathbb{C}) \setminus \widehat{M} \\ f & \text{on } D \times \Delta \end{cases}.$$

Observe that any connected component of $((U \cap V) \times \mathbb{C}) \setminus \widehat{M}$ is a domain intersecting $D \times \Delta$. Thus, from the identity principle, the functions \widehat{f}_U and \widehat{f}_V agree also on $((U \cap V) \times \mathbb{C}) \setminus \widehat{M}$. Hence, \widehat{f} is well defined and holomorphic on $(D \times \Delta) \cup \cup(((U \cap V) \times \mathbb{C}) \setminus \widehat{M})$.

Step 3.

Since D is σ -compact, there exists a countable covering $\{U_j\}_{j=1}^\infty$ such that each U_j is biholomorphic to a domain in \mathbb{C}^n . Because A is not pluripolar, there exists j_0 such that $A \cap U_{j_0}$ is not pluripolar. From Step 1, we obtain a relatively closed pluripolar set $\widehat{M}_{U_{j_0}} \subset U_{j_0} \times \mathbb{C}$ having properties (1) to (4). From Step 2, for any $U_j \neq U_{j_0}$ such that $U_j \cap U_{j_0} \neq \emptyset$ there exists a relatively closed pluripolar set $\widehat{M}_{U_j} \subset U_j \times \mathbb{C}$ having same properties. Define $D_1 := \bigcup \{U_j : U_j \cap U_{j_0} \neq \emptyset\}$, a set $\widehat{M}_{D_1} := \bigcup \{\widehat{M}_{U_j} : U_j \in D_1\}$ and a function

$$\widehat{f}_{D_1} := \begin{cases} \widehat{f}_{U_j} & \text{on } (U_j \times \mathbb{C}) \setminus \widehat{M}_{D_1}, j \neq j_0 \\ \widehat{f}_{U_{j_0}} & \text{on } (U_{j_0} \times \mathbb{C}) \setminus \widehat{M}_{D_1} \\ f & \text{on } D \times \Delta \end{cases}$$

From Step 2, \widehat{f}_{D_1} is well defined and holomorphic on $(D \times \Delta) \cup ((D_1 \times \mathbb{C}) \setminus \widehat{M}_{D_1})$ and $\widehat{f}_{D_1} = f$ on $D \times \Delta$. Thus, \widehat{M}_{D_1} has properties (1) to (4). Now, for any $U_j \notin D_1$ such that $U_j \cap D_1 \neq \emptyset$ from Step 2 there exists a relatively closed pluripolar set $\widehat{M}_{U_j} \subset U_j \times \mathbb{C}$ having properties (1) to (4). Define $D_2 := \bigcup \{U_j : U_j \cap D_1 \neq \emptyset\}$, $\widehat{M}_{D_2} := \bigcup \{\widehat{M}_{U_j} : U_j \in D_2\}$ and

$$f_{D_2} := \begin{cases} \widehat{f}_{U_j} & \text{on } (U_j \times \mathbb{C}) \setminus \widehat{M}_{D_2}, U_j \notin D_1 \\ \widehat{f}_{D_1} & \text{on } (D_1 \times \mathbb{C}) \setminus \widehat{M}_{D_2} \\ f & \text{on } D \times \Delta \end{cases}$$

Once again, \widehat{f}_{D_2} is well defined and holomorphic on $(D \times \Delta) \cup (D_2 \times \mathbb{C}) \setminus \widehat{M}_{D_2}$, $\widehat{f}_{D_2} = f$ on $D \times \Delta$ and \widehat{M}_{D_2} has all properties (1) to (4).

We obtain an open covering $\{D_k\}_{k=1}^{\infty}$ of D such that for each $k=1,2,\dots$ there exists a relatively closed pluripolar set \widehat{M}_{D_k} having properties (1) to (4). Define $\widehat{M} := \bigcap_{k=1}^{\infty} \widehat{M}_{D_k}$ and a function $\widehat{f} := \widehat{f}_{D_k}$ on $(D_k \times \mathbb{C}) \setminus \widehat{M}$. Thus, \widehat{f} is well defined and holomorphic on $(D \times \mathbb{C}) \setminus \widehat{M}$ and $\widehat{f} = f$ on $D \times \Delta$. Hence, the set \widehat{M} has all properties (a) to (d) and proof of Theorem 2.1 is finished. \square

References

- [1] Chirka E.M., Sadullaev A., *On continuation of functions with polar singularities*, Mat. Sb. (N.S.), Vol. 132(174), 1987, 383-390.
- [2] Jarnicki M., Pflug P., *An extension theorem for separately holomorphic functions with pluripolar singularities*, Trans. Amer. Math. Soc. 355, 2003, 1251-1267.
- [3] Jarnicki M., Pflug P., *A general cross theorem with singularities*, Analysis Munich 27, 2007, 181-212.
- [4] Jarnicki M., Pflug P., *A new cross theorem for separately holomorphic functions*, Proc. Amer. Math. Soc. 138, 2010, 3923-3932.
- [5] Jarnicki M., Pflug P., *Separately analytic functions*, EMS Publishing House, 2011.
- [6] Nguyen V.-A., Pflug P., *Cross theorems with singularities*, J. Geom. Anal. 20, 2010, 193-218.
- [7] Öktem O., *Extension of separately analytic functions and applications to range characterization of exponential Radon transform*, Ann. Polon. Math., 70, 1998, 195-213.
- [8] Öktem O., *Extension of separately analytic functions and applications to mathematical tomography*, Thesis, Departament of Mathematics, Stockholm University, 1999.
- [9] Zającka M., *Twierdzenia typu Hartogsa*, Thesis, Faculty of Mathematics and Computer Science, Jagiellonian University, Kraków 2013.

MAŁGORZATA ZAJĘCKA*

A NOTE ON SINGULARITY OF FIBERS OF SINGULAR SETS

UWAGA O SINGULARNOŚCI WŁÓKIEN ZBIORÓW SINGULARNYCH

Abstract

The paper presents a general theorem on fibers of singular sets: Let D_1 be a connected σ -compact Josefson manifold and let D_2 be a σ -compact complex manifold. Let $\Omega \subset D_1 \times D_2$ be a domain and let $\Omega \subset M$ be a singular set with respect to the family $\mathcal{F} \subset \mathcal{O}(\Omega \setminus M)$ such that the set $\{a_1 \in D_1 : \text{the fiber } M_{(a_1, \cdot)} \text{ is not pluripolar}\}$ is pluripolar in D_1 . It is shown that there exists a pluripolar set $Q \subset D_1$ such that for every $a_1 \in \pi_{D_1}(\Omega) \setminus Q$, the fiber $M_{(a_1, \cdot)}$ is singular in $\Omega_{(a_1, \cdot)}$ with respect to the family $\mathcal{F}_a := \{f(a_1, \cdot) : f \in \mathcal{F}\} \subset \mathcal{O}(\Omega_{(a_1, \cdot)})$.

Keywords: singular set, fiber of singular set, pluripolar set, complex manifold

Streszczenie

W artykule przedstawiono dowód ogólnego twierdzenia dotyczącego własności włókien zbiorów singularnych: Niech D_1 będzie spójną, σ -zwartą rozmaiłością Josefszona oraz niech D_2 będzie σ -zwartą rozmaiłością zespoloną. Niech $\Omega \subset D_1 \times D_2$ będzie obszarem oraz niech $\Omega \subset M$ będzie zbiorem singularnym względem rodziny $\mathcal{F} \subset \mathcal{O}(\Omega \setminus M)$, takim, że zbiór $\{a_1 \in D_1 : \text{włókno } M_{(a_1, \cdot)} \text{ nie jest pluripolarne}\}$ jest pluripolarny w D_1 . Wykazano, że istnieje wtedy zbiór pluripolarny $Q \subset D_1$ taki, że dla dowolnego $a_1 \in \pi_{D_1}(\Omega) \setminus Q$ włókno $M_{(a_1, \cdot)}$ jest singularne w $\Omega_{(a_1, \cdot)}$ względem rodziny $\mathcal{F}_a := \{f(a_1, \cdot) : f \in \mathcal{F}\} \subset \mathcal{O}(\Omega_{(a_1, \cdot)})$.

Słowa kluczowe: zbiór singularny, włókno zbioru singularnego, zbiór pluripolarny, rozmaiłość zespolona

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1. Introduction and prerequisites

In [2], the authors proved a result which states that almost all sections of Riemann domains of holomorphy are regions of holomorphy.

Let (X, p) be a Riemann domain over $\mathbb{C}^n = \mathbb{C}^k \times \mathbb{C}^l$, where $p = (u, v): X \rightarrow \mathbb{C}^k \times \mathbb{C}^l$, and define $D_k := u(X)$. Let $\emptyset \neq \mathcal{F} \subset \mathcal{O}(X)$. For $a \in D_k$, define a fiber $X_a := u^{-1}(a)$, a function $p_a := v|_{X_a}$, and a family $\mathcal{F}_a := \{f|_{X_a} : f \in \mathcal{F}\}$.

Theorem 1.1. (Theorem 2.2 from [2]) Let $\emptyset \neq \mathcal{F} \subset \mathcal{O}(X)$ and assume that (X, p) is an \mathcal{F} -domain of holomorphy. Then there exists a pluripolar set $S_k \subset D_k$ such that for every $a \in D_k \setminus S_k$, (X_a, p_a) is an \mathcal{F}_a -region of holomorphy.

Remark 1.2. Theorem 1.1 remains true if we assume that (X, p) is a countable at infinity \mathcal{F} -region of holomorphy (see Theorem 9.1.2 in [3]).

Now, recall the definition of a singular set. For an n -dimensional complex manifold X , let M be a closed subset of X such that for any domain $\Omega \subset X$, the set $\Omega \setminus M$ is connected and dense in Ω (for instance, let M be a pluripolar set). Let \mathcal{F} be a family of functions holomorphic on $X \setminus M$.

Definition 1.3. A point $a \in M$ is called *singular with respect to the family \mathcal{F}* , if for any open connected neighborhood U_a of the point a , there exists a function $f \in \mathcal{F}$, that does not extend holomorphically on U_a . We call M *singular with respect to the family \mathcal{F}* , if every point $a \in M$ is singular with respect to \mathcal{F} .

A consequence of Theorem 1.1 is a similar property of fibers of singular sets in the Riemann regions of holomorphy from [3].

Theorem 1.4. (Proposition 9.1.4 from [3], see also Lemma 3.3 from [2]) Let (D, p_D) and (G, p_G) be Riemann domains over \mathbb{C}^k and \mathbb{C}^l , respectively. Let $\Omega \subset D \times G$ be a Riemann region of holomorphy and let $M \subset \Omega$ be a relatively closed pluripolar set that is singular with respect to a family $S \subset \mathcal{O}(\Omega \setminus M)$. There then exists a pluripolar set $P \subset D$ such that for any $a \in \pi_D(\Omega) \setminus P$ the fiber $M_{(a,\cdot)} := \{b \in G : (a, b) \in M\}$ is singular with respect to the family $S_a := \{f(a, \cdot) : f \in S\} \subset \mathcal{O}(\Omega_{(a,\cdot)} \setminus M_{(a,\cdot)})$, where $\pi_D(\Omega)$ denotes the projection of Ω to D .

Following the proof of Theorem 1.4, it becomes clear that we can replace the assumption of M being relatively closed and pluripolar by a weaker assumption: we need only that the set $\{a \in D : \text{the fiber } M_{(a,\cdot)} \text{ is not pluripolar}\}$ is pluripolar.

Theorem 1.4 proved to be one of the key properties used in the theory of extensions of functions separately holomorphic on different kinds of objects called crosses. This topic has a long history in complex analysis (for the details of its evolution, see the introduction to [3]) and was started by W.F. Osgood and F. Hartogs with the famous theorem stating that every separately holomorphic function is, in fact, holomorphic ([1, 5]). One of the latest and

most general and technically demanding results in the case of crosses with singularities on Riemann domains (see Theorem 3.2 in [2], Theorem 10.2.9 in [3], Main Theorem in [6]) are obtained using, among other strong tools, Theorem 1.4.

Recently, the context of cross theory has moved from Riemann domains to more general objects, such as complex manifolds or even analytic spaces (see [4]). However, the case of extensions on crosses with singularities on complex manifolds still remains open, partially because of a lack of necessary base results which were available for Riemann domains.

In this paper, we show proof of generalisation of Jarnicki and Pflug result which is one of the main tools needed to build a theory of crosses with singularities on complex manifolds. Since the proof of original Theorem 1.1 (and thus the proof of Theorem 1.4) was based on strong results, it is surprising that the proof of the main theorem presented in the next section is so elementary.

Main Theorem. Let D_1 be a connected σ -compact Josefson manifold (i.e. D_1 is a countable at infinity complex manifold such that every locally pluripolar set in D_1 is globally pluripolar) of dimension n_1 and let D_2 be a σ -compact complex manifold of dimension n_2 . Let $\Omega \subset D_1 \times D_2$ be a domain and let $M \subset \Omega$ be a singular set with respect to the family $\mathcal{F} \subset \mathcal{O}(\Omega \setminus M)$ such that the set $\{a_1 \in D_1 : \text{the fiber } M_{(a_1, \cdot)} \text{ is not pluripolar}\}$ is pluripolar in D_1 . Then there exists a pluripolar set $Q \subset D_1$ such that for every $a_1 \in \pi_{D_1}(\Omega) \setminus Q$, the fiber $M_{(a_1, \cdot)}$ is singular in $\Omega_{(a_1, \cdot)}$ with respect to the family $\mathcal{F}_a := \{f(a_1, \cdot) : f \in \mathcal{F}\} \subset \mathcal{O}(\Omega_{(a_1, \cdot)})$, where $\pi_{D_1}(\Omega)$ denotes the projection of Ω to D_1 and for $B \subset D_1 \times D_2$ and $a_1 \in D_1$, we put $B_{(a_1, \cdot)} := \{a_2 \in D_2 : a = (a_1, a_2) \in B\}$.

2. Proof of Main Theorem

Fix $a = (a_1, a_2) \in M$, where $a_1 \in D_1$, $a_2 \in D_2$. Let $\Phi_j : U_j \rightarrow \widetilde{U}_j$ be a biholomorphic mapping such that U_j is an open neighbourhood of a_j , \widetilde{U}_j is an Euclidean ball in \mathbb{C}^{n_j} , $\Phi_j(a_j) = 0$, $j = 1, 2$, and $U_a := U_1 \times U_2 \subset \Omega$.

Define $\Phi := (\Phi_1, \Phi_2)$ and $N := \Phi(M \cap U_a)$, $\mathcal{F}_a := \{f|_{U_a} : f \in \mathcal{F}\}$, $\widetilde{\mathcal{F}}_a := \{f \circ \Phi^{-1} : f \in \mathcal{F}_a\}$. Then N is a relatively closed subset of $\widetilde{U} := \widetilde{U}_1 \times \widetilde{U}_2$ and $\widetilde{\mathcal{F}}_a \subset \mathcal{O}(\widetilde{U} \setminus N)$. We show that N is singular with respect to the family $\widetilde{\mathcal{F}}_a$.

Fix a $w \in N$ and define $b := \Phi^{-1}(w) \in M$. Assume that there exists an open neighborhood \widetilde{V}_w of w such that every function $\tilde{f} \in \widetilde{\mathcal{F}}_a$ extends holomorphically on \widetilde{V}_w . Let $V_b := \Phi^{-1}(\widetilde{V}_w)$. Fix $f \in \mathcal{F}_a$ and define $\tilde{f} := f \circ \Phi^{-1} \in \widetilde{\mathcal{F}}_a$. Then \tilde{f} extends

to a function \widetilde{F} holomorphic on \widetilde{V}_w . Define $F := F \circ \Phi|_{V_b} \in \mathcal{O}(V_b)$. Since $F = \widetilde{F} \circ \Phi = \widetilde{f} \circ \Phi = f$ on the nonempty open set $V_b \setminus M$, we conclude that F is a holomorphic extension of f to V_b – a contradiction.

Now, from Theorem 1.4, there exists a pluripolar set $\widetilde{Q}_a \subset \mathbb{C}^{n_1}$ such that for any $w_1 \in \widetilde{U}_1 \setminus \widetilde{Q}_a$ the fiber $N_{(w_1, \cdot)} := \{w_2 \in \mathbb{C}^{n_2} : (w_1, w_2) \in N\}$ is singular with respect to the family $\widetilde{\mathcal{F}}_{w_1} := \{\widetilde{f}(w_1, \cdot) : \widetilde{f} \in \widetilde{\mathcal{F}}_a\}$. Define $b_1 := (\Phi_1)^{-1}(w_1) \in U_1$, $Q_a := (\Phi_1)^{-1}(\widetilde{Q}_a)$. Then Q_a is pluripolar in D_1 and

$$N_{(w_1, \cdot)} = \{w_2 \in \mathbb{C}^{n_2} : \exists b_2 \in U_2 : \Phi_2(b_2) = w_2, (b_1, b_2) \in M\} = \Phi_2(M_{(b_1, \cdot)}).$$

Using similar reasoning as before, we show that for any $b_1 \in U_1 \setminus Q_a$ the fiber $M_{(b_1, \cdot)}$ is singular with respect to the family $\mathcal{F}_{b_1} = \{f(b_1, \cdot) : f \in \mathcal{F}_a\}$.

From $\{U_a\}_{a \in M}$, we choose a countable covering $\{U_{a_j}\}_{j=1}^\infty$ of the set M . Define $Q := \bigcup_{j=1}^\infty Q_{a_j} \cup \{b_1 \in D_1 : \text{the fiber } M_{(b_1, \cdot)} \text{ is not pluripolar}\}$.

Because D_1 is a Josefson manifold, Q is pluripolar in D_1 . We show that for any $b_1 \in \pi_{D_1}(\Omega) \setminus Q$, the fiber $M_{(b_1, \cdot)}$ is singular with respect to the family $\mathcal{F}_{b_1} := \{f(b_1, \cdot) : f \in \mathcal{F}\}$.

Fix $b_1 \in \pi_{D_1}(\Omega) \setminus Q$, $b_2 \in M_{(b_1, \cdot)}$. Assume that there exists an open neighbourhood V_{b_2} of b_2 such that any function $f(b_1, \cdot), f \in \mathcal{F}$, extends holomorphically on V_{b_2} . Fix $f \in \mathcal{F}$. Because $(b_1, b_2) \in M$, then there exists a_j , $j \in \{1, 2, \dots\}$, such that $(b_1, b_2) \in U_{a_j} = U_{1,j} \times U_{2,j}$. Thus $b_1 \in U_{1,j} \setminus Q_{a_j}$ and $f|_{U_{a_j}}(b_1, \cdot)$ extends holomorphically on $(V_{b_2} \cap U_{2,j})$, but we already know that the fiber $M_{(b_1, \cdot)}$ is singular with respect to the family $\{f(b_1, \cdot) : f \in \mathcal{F}_{a_j}\} = \{f|_{U_{a_j}}(b_1, \cdot) : f \in \mathcal{F}\}$ – a contradiction.

References

- [1] Hartogs F., *Zur Theorie der analytischen Funktionen mehrerer unabhängiger Veränderlichen, insbesondere über die Darstellung derselben durch Reihen, welche nach Potenzen einer Veränderlichen fortschreiten*, Math. Ann., 62, 1906, 1-88.
- [2] Jarnicki M., Pflug P., *A remark on separate holomorphy*, Studia Mat. 174, 2006, 309-317.
- [3] Jarnicki M., Pflug P., *Separately analytic functions*, EMS Publishing House, 2011.
- [4] Lewandowski A., *A remark on the relative extremal function*, to appear.
- [5] Osgood W.F., *Note über analytische Functionen mehrerer Veränderlichen*, Math. Annalen 52, 1899, 462-464.
- [6] Zajęcka M., *A Hartogs type extension theorem for generalized (N, k) -crosses with pluripolar singularities*, Colloq. Math. 127, 2012, 143-160.

INFORMATICS

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KINETIC PROPERTIES AND CHARGE TRANSPORT PHYSICS OF ISOTROPIC SEMICONDUCTOR CRYSTALS

WŁASNOŚCI KINETYCZNE I FIZYKA TRANSPORTU ŁADUNKU W IZOTROPOWYCH KRYSZTAŁACH PÓŁPRZEWODNIKOWYCH

Abstract

This paper presents general formulas for the kinetic properties of semiconductor crystals expressed in terms of the Fermi integrals. The formulas provide algorithms for the calculation or identification of the kinetic properties and constitute (together with the results of theoretical analysis and other presented relationships) the mathematical model of the charge carrier transport phenomena in semiconductor crystals.

Keywords: kinetic phenomena, charge carriers, scattering mechanisms, semiconductor crystals

Streszczenie

Uzyskane i przedstawione w artykule ogólne wzory opisujące własności kinetyczne kryształów półprzewodnikowych są wyrażone poprzez całki Fermiego. Te wzory są podstawą algorytmu do obliczenia i identyfikacji tych własności, stanowiąc (łącznie z wynikami analizy teoretycznej i innymi przedstawionymi zależnościami) model matematyczny zjawisk transportu nośników ładunku elektrycznego w kryształach półprzewodnikowych.

Słowa kluczowe: zjawiska kinetyczne, nośniki ładunku, mechanizmy rozpraszania, kryształy półprzewodnikowe

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1. Introduction

The theoretical problems of how to determine the kinetic properties of a semiconductor crystal are connected mainly with the following problems:

- determining the energy spectrum $\varepsilon(\overline{p})$,
- determining the scattering function $u(\varepsilon)$,
- determining the reduced chemical potential μ^* .

In this paper, a theoretical analysis of the kinetic properties of the semiconductor crystals is presented. The kinetic properties of the semiconductor crystals are considered in the framework of the statistical theory of the solid state. This discussion is based on general formulas which are expressed in terms of the Fermi integrals. These formulas provided algorithms for the calculation of the kinetic properties and, together with other relations, constitute the mathematical model of the charge carrier transport phenomena in semiconductor crystals.

The calculation or identification of the kinetic properties of a crystal and the prediction of its useful technological properties are closely related [1, 2, 4, 8, 10, 13].

2. Theoretical analysis of the kinetic properties of semiconductor crystals

According to the statistical theory of the solid state, in the presence of a weak magnetic field, or when the magnetic field is absent, the kinetic properties of isotropic doped semiconductor crystals may be described by general formulas – these are expressed in terms of the Fermi integrals.

Some of these properties are as follows:

- the resistivity $\rho(T, \mu^*)$
- the Hall coefficient $R(T, \mu^*)$,
- the thermopower (the Seebeck coefficient) $\alpha(T, \mu^*)$,
- the Nernst – Ettingshausen constant $N(T, \mu^*)$,
- the thermal conductivity $\chi(T, \mu^*)$,
- the conductivity $\sigma(T, \mu^*)$,
- the carrier density $n(T, \mu^*)$,
- the carrier mobility $U_D(T, \mu^*)$,
- the Hall mobility $U_H(T, \mu^*)$,

and the appropriate formulas based on the Fermi integrals take the forms:

$$\rho(T, \mu^*) = \frac{1}{en} \frac{J00(T, \mu^*)}{J01(T, \mu^*)}, \quad (1)$$

$$R(T, \mu^*) = \frac{1}{zen} \frac{J00(T, \mu^*)J02(T, \mu^*)}{J01(T, \mu^*)^2}, \quad (2)$$

$$\alpha(T, \mu^*) = \frac{k}{ze} \left(\frac{J11(T, \mu^*)}{J01(T, \mu^*)} - \frac{\mu}{kT} \right), \quad (3)$$

$$N(T, \mu^*) = \frac{k}{e} \left| \frac{R(T, \mu^*)}{\rho(T, \mu^*)} \right| \left(\frac{J11(T, \mu^*)}{J01(T, \mu^*)} - \frac{J12(T, \mu^*)}{J02(T, \mu^*)} \right), \quad (4)$$

$$\chi(T, \mu^*) = \left(\frac{k}{e} \right)^2 \frac{T}{\rho(T, \mu^*)} \left(\frac{J21(T, \mu^*)}{J01(T, \mu^*)} - \left(\frac{J11(T, \mu^*)}{J01(T, \mu^*)} \right)^2 \right), \quad (5)$$

$$\sigma(T, \mu^*) = en(T, \mu^*) \frac{J01(T, \mu^*)}{J00(T, \mu^*)} = en(T, \mu^*) U_D(T, \mu^*), \quad (6)$$

$$n(T, \mu^*) = J00(T, \mu^*), \quad (7)$$

$$U_D(T, \mu^*) = \frac{J01(T, \mu^*)}{J00(T, \mu^*)}, \quad (8)$$

$$U_H(T, \mu^*) = \left| \frac{R}{\rho} \right| = \frac{J02(T, \mu^*)}{J01(T, \mu^*)}. \quad (9)$$

In the above expressions for the kinetic properties of semiconductor crystals, the symbols $J_{ij}(T, \mu^*)$ are reserved for the Fermi integrals (the kinetic integrals) given by the following formulas:

$$J_{ij}(T, \mu^*) = \int_0^\infty \left(\frac{\varepsilon}{kT} \right)^i u^j G(\varepsilon) \left(-\frac{df_0}{d\varepsilon} \right) d\varepsilon, \quad J00(T, \mu^*) = \int_0^\infty G(\varepsilon) \left(-\frac{df_0}{d\varepsilon} \right) d\varepsilon. \quad (10)$$

The analysis of the kinetic integrals (10) shows us that the following inequality takes place [6]:

$$\left(\frac{J01(T, \mu^*)}{J00(T, \mu^*)} \right)^2 \leq \left(\frac{J02(T, \mu^*)}{J00(T, \mu^*)} \right).$$

From this inequality, it follows that:

$$\frac{J00(T, \mu^*) J02(T, \mu^*)}{J01(T, \mu^*)^2} = f_R(T, \mu^*) \geq 1.$$

Commonly, the function $f_R(T, \mu^*)$ in lecture courses on solid state and semiconductor physics is called the Hall factor [7, 8]. It is easy to show that this function has a minimum value equal to 1 in a semiconductor crystal, where the current carriers are degenerated.

Using the function (10), formulas (8) and (9) describe the carrier and Hall mobilities:

$$U_D(T, \mu^*) = \frac{J01(T, \mu^*)}{J00(T, \mu^*)},$$

$$U_H(T, \mu^*) = \frac{J02(T, \mu^*)}{J01(T, \mu^*)}.$$

Thus, we have:

$$\frac{U_H(T, \mu^*)}{U_D(T, \mu^*)} = \frac{J00(T, \mu^*)J02(T, \mu^*)}{J01(T, \mu^*)^2} = f_R(T, \mu^*) \geq 1.$$

Consequently, we also have $U_H(T, \mu^*) = U_D(T, \mu^*)f_R(T, \mu^*)$, that is $U_H(T, \mu^*) > U_D(T, \mu^*)$.

From definition of the carrier mobility $U_D(T, \mu^*)$, there is:

$$U_D(T, \mu^*) = \frac{J01(T, \mu^*)}{J00(T, \mu^*)} = \frac{\int_0^\infty u(\varepsilon)G(\varepsilon)\left(-\frac{df_0}{d\varepsilon}\right)d\varepsilon}{\int_0^\infty G(\varepsilon)\left(-\frac{df_0}{d\varepsilon}\right)d\varepsilon}.$$

In statistical physics, this expression has mathematical sense of the average value of the function $u(\varepsilon)$, for all values ε , $\varepsilon \in [0, \infty)$ and this value is denoted by $\langle \dots \rangle$. Clearly, in statistical physics, the carrier mobility $U_D(T, \mu^*)$ is given by:

$$U_D(T, \mu^*) = \langle u(\varepsilon) \rangle_{(T, \mu^*)},$$

where the subscripts (T, μ^*) denote that the average values $\langle u(\varepsilon) \rangle$ are analytically dependent of the chemical potential and temperature. Hence, the carrier mobility $U_D(T, \mu^*)$ equals the average value of the function $u(\varepsilon)$ for all values of ε .

We will now consider the Hall mobility which by formula (9) is given as:

$$U_H(T, \mu^*) = \frac{J02(T, \mu^*)}{J01(T, \mu^*)} = \frac{J02(T, \mu^*)J00(T, \mu^*)}{J00(T, \mu^*)J01(T, \mu^*)} = \frac{\langle u(\varepsilon)^2 \rangle_{(T, \mu^*)}}{\langle u(\varepsilon) \rangle_{(T, \mu^*)}}.$$

The Hall factor can be expressed in terms of the averages of the scattering function $u(\varepsilon)$ ($u(\varepsilon)$ is the average velocity of carriers in the presence of the unit electric field \vec{E}) in the following way:

$$f_R(T, \mu^*) = \frac{J00(T, \mu^*)J02(T, \mu^*)}{J01(T, \mu^*)^2} = \left(\frac{J00(T, \mu^*)}{J01(T, \mu^*)} \right)^2 \frac{J02(T, \mu^*)}{J00(T, \mu^*)} = \frac{\langle u(\varepsilon)^2 \rangle_{(T, \mu^*)}}{\langle u(\varepsilon) \rangle_{(T, \mu^*)}^2}.$$

In statistical physics is proved that $\langle u(\varepsilon)^2 \rangle_{(T, \mu^*)} \geq \langle u(\varepsilon) \rangle_{(T, \mu^*)}^2$, then we obtain $f_R(T, \mu^*) \geq 1$.

Analysis of formulas (1)–(10) for the kinetic properties of semiconductor crystals that is based on the above calculations shows that these kinetic properties analytically depend on the average values $\langle u(\varepsilon) \rangle_{(T, \mu^*)}$ and $\langle u(\varepsilon) \rangle_{(T, \mu^*)}^2$.

Carriers in semiconductor crystals are scattered by defects in the crystals hence the concept or notion of the scattering function $u(\varepsilon)$ which describes the effect of

these scattering processes on kinetic properties of these crystals is a serious problem in semiconductor and solid state physics.

Formula (10) shows that the kinetic integrals $J_{ij}(T, \mu^*)$ depend on the Fermi-Dirac equilibrium distribution function

$$f_0(\varepsilon, \mu^*) = \left(\exp\left(\frac{\varepsilon}{kT} - \mu^*\right) + 1 \right)^{-1}, \quad \mu^* = \frac{\mu}{kT}. \quad (11)$$

From this formula, and from formulas (1)–(10), it follows that all properties of the crystal, which are described by formulas (1)–(10), analytically depend on the reduced chemical potential μ^* .

There is also an analytical dependence of kinetic properties on $G(\varepsilon) = \int_0^\varepsilon g(\varepsilon) d\varepsilon$ [8] in these formulae. Here, $g(\varepsilon)$ is the density of states (DOS) lying in allowed bands which depends on the analytical form of the energy spectrum $\varepsilon_{\vec{p}} = \varepsilon(\vec{p})$.

In isotropic crystals $\varepsilon_{\vec{p}} = \varepsilon(|\vec{p}|) = \varepsilon(p)$, where $p = \sqrt{p_x^2 + p_y^2 + p_z^2}$. Thus, all kinetic properties of these crystals which are described by formulas (1)–(10) analytically depend on the energy spectrum $\varepsilon_{\vec{p}} = \varepsilon(p)$.

Formulas (1)–(10) show that with the exception of the density $n(T, \mu^*)$ of current carriers in a semiconductor (formula (7)), all kinetic properties of crystals explicitly depend on the scattering function $u(\varepsilon)$ – this describes the effect of these scattering processes on the kinetic properties of these crystals [8, 9]. Computations of values of the scattering function $u(\varepsilon)$ are connected with problems on determination the energy spectrum $\varepsilon(\vec{p})$ and scattering probability $W(\vec{p}, \vec{p}')$. The explicit form of this probability depends on the nature of the semiconductor crystal and the type of defects in this crystal by which free charge carriers are scattered [1, 10].

3. The energy spectrum and the discrete density of states (DOS) of the charge carrier energy in a semiconductor crystal

In the quantum theory of crystal structure, the energy spectrum is called the function – this describes the dependence of the total charge carrier energy ε on this carrier quasimomentum \vec{p} in this crystal, that is $\varepsilon_{\vec{p}} = \varepsilon(\vec{p})$ [8].

According to quantum mechanics, the charge carrier quasimomentum \vec{p} in crystals varies discretely and there are as many values of \vec{p} as there are structure particles in the crystal. In this connection, it should be noted the charge carrier energy is quantised. Moreover, it was proved [7] that the energy spectrum has symmetry properties being a periodic and

even function of the vector \vec{p} : $\varepsilon(\vec{p}) = \varepsilon(-\vec{p})$, $\varepsilon(\vec{p}) = \varepsilon(-\vec{p} + \vec{P})$, where $\vec{P} = \hbar \vec{K}_n$. In this relationship, \hbar is the Planck's constant, \vec{K}_n is the reciprocal lattice vector.

A free charge carrier with energy $\varepsilon_{\vec{p}} = \varepsilon(\vec{p})$ in crystals has the velocity of chaotic motion $\vec{v}_{\vec{p}} = \nabla_{\vec{p}} \varepsilon(\vec{p})$ and this carrier subjected to the action of a force \vec{F} will acquire an acceleration $\vec{a}_{\vec{p}}$, which is defined by formula [8]:

$$\vec{a}_{\vec{p}} = \left(\frac{\partial^2 \varepsilon(\vec{p})}{\partial p_i \partial p_k} \right) \vec{F} = \left(\frac{1}{m_{ik}(\vec{p})} \right) \vec{F},$$

where the tensor $\left(\frac{\partial^2 \varepsilon(\vec{p})}{\partial p_i \partial p_k} \right) = \left(\frac{1}{m_{ik}(\vec{p})} \right)$ is called the reciprocal effective mass tensor. Once

the symmetry properties of the energy spectrum are known, it is easy to approximate this energy spectrum of the free energy carriers in a semiconductor crystal.

It can be seen from these calculations that in the case of anisotropic crystals, the energy spectrum is described by the constant energy ellipsoid S , having the following anisotropic parabolic form:

$$\varepsilon_{\vec{p}} \cong \frac{p_1^2}{2m_{11}} + \frac{p_2^2}{2m_{22}} + \frac{p_3^2}{2m_{33}},$$

and in isotropic crystals, it is the constant energy sphere S , having the isotropic parabolic form:

$$\varepsilon_{\vec{p}} \cong \frac{p_1^2 + p_2^2 + p_3^2}{2m} = \frac{p^2}{2m},$$

where m_{11} , m_{22} , m_{33} are the diagonal components of the charge carrier effective mass tensor [8].

In quantum mechanics, the quantity $G(\varepsilon)$, which determines the number of states of particles in a crystal, with the energy range between $\varepsilon=0$ and $\varepsilon(\vec{p})$, is given by the well-known general expression:

$$G(\varepsilon) = \int_0^\varepsilon d\varepsilon \frac{2}{h^3} \oint_S \frac{dS}{|\nabla_{\vec{p}} \varepsilon_{\vec{p}}|}, \quad S = \{(\vec{p}, \varepsilon) : \varepsilon_{\vec{p}} = \varepsilon(\vec{p}), \varepsilon = \text{const}\}. \quad (12)$$

From the definition of the density of the energy states (DOS), we obtain:

$$g(\varepsilon) = \frac{dG(\varepsilon)}{d\varepsilon}. \quad (13)$$

Substituting the value $G(\epsilon)$ into equation (13), we have:

$$g(\epsilon) = \frac{2}{h^3} \oint_S \frac{dS}{|\nabla_{\vec{p}} \epsilon_{\vec{p}}|}, \quad (14)$$

where the integral has to be evaluated for the constant energy surface S , which is defined by the energy spectrum $\epsilon_{\vec{p}} = \epsilon(\vec{p})$.

Since $g(\epsilon) = \frac{dG(\epsilon)}{d\epsilon}$, then $G(\epsilon) = \int_0^\epsilon g(\epsilon) d\epsilon$.

It is clear from formulas (12) and (14), that to determine the density of state (DOS), the constant-energy surface S must be known.

4. The calculation of the kinetic properties of isotropic doped semiconductor crystals

The presented formulas provide algorithms for the calculation of the kinetic properties (1)–(9) of isotropic doped semiconductor crystals.

In isotropic crystals, the energy spectrum is isotropic and it takes the form: $\epsilon_p = \epsilon(p)$, where $p = |\vec{p}| = \sqrt{p_1^2 + p_2^2 + p_3^2}$. It can then also be rewritten in the form: $p = p(\epsilon)$.

We shall use formulae (12) and (14) to determine the quantities $G(\epsilon)$ and $g(\epsilon)$, taking into consideration the crystal with the isotropic energy spectrum. Now, the quantities $G(\epsilon)$ and $g(\epsilon)$ have the following forms:

$$G(\epsilon) = \frac{8\pi}{3h^3} p^3(\epsilon), \quad (15)$$

$$g(\epsilon) = \frac{dG(\epsilon)}{d\epsilon} = \frac{8\pi}{3h^3} p^2(\epsilon) \frac{dp(\epsilon)}{d\epsilon}. \quad (16)$$

In the above formulae, the function $p = p(\epsilon)$ is defined by the isotropic energy spectrum $\epsilon_p = \epsilon(p)$.

There are the different modes of the charge carriers scattering in semiconductors [1, 8, 9, 13]. Here, the main mechanisms of interest are scattering by the acoustic phonons, the optical phonons, the point defects in semiconductors, and the charged impurities (the ions).

The scattering function of the main scattering modes is given by [1, 2, 3, 10]:

$$u(\epsilon) = U^{(r)}(T) p^{(2r-3)} \left(\frac{d\epsilon}{dp} \right)^2, \quad (17)$$

where $U^{(r)}(T)$ is the temperature dependent function known for the actual scattering mode, r is the parameter and it has the following values [1]:

- $r = 0$ for the scattering by the acoustic phonons and the point defects in semiconductors.
- $r = 1$ for the scattering by the optical phonons when the crystal's temperature is larger than the Debay temperature Θ .
- $r = 2$ for the scattering by the charged impurities (the ions) in the crystal lattice.

In the case of the combined scattering mode, the general scattering function u is given by the formula:

$$\frac{1}{u} = \frac{1}{u(r_1)} + \frac{1}{u(r_2)} + \dots + \frac{1}{u(r_n)}, \quad (18)$$

where $\frac{1}{u(r_i)}$ is the scattering function of the simple scattering mode with the parameter r_i , $i = 0, \dots, 2$.

Now, we can use the formulae (1)–(9) to the calculation of the kinetic properties of isotropic doped semiconductor crystals with a narrow energy gap E_g .

In these crystals, the electrons and holes energy spectra are described by the nonparabolic Kane's band [1, 5]:

$$\frac{p^2}{2m^*} = \varepsilon + \frac{\varepsilon^2}{E_g}, \quad \text{that is} \quad p = \sqrt{2m^* \varepsilon \left(1 + \frac{\varepsilon}{E_g}\right)}, \quad (19)$$

where m^* is the parameter of the spectrum which plays a role of the electron or hole effective mass, E_g is the energy gap width.

For the energy spectrum (19) and the actual scattering modes (which are defined by the parameter r in formula (17)) taking into consideration formulae (15) and (17), the kinetic integral (10) takes form:

$$J_{ij}(T, \mu^*) = [U^{(r)}(T)]^j N_c(T) I_{ij}(T, \mu^*). \quad (20)$$

In formula (20), the following notations were used:

$N_c(T)$ is the conduction band effective density of states and is defined by [8]:

$$N_c(T) = \frac{8}{3\sqrt{\pi}} \left(\frac{2m^* kT}{h^2} \right)^{3/2}. \quad (21)$$

$U^{(r)}(T)$ is the temperature dependent function, which has dimension of the mobility and is defined by [1, 3, 10]:

$$U^{(r)}(T) = \left[\frac{1}{T} (U_A \delta(0, r) + U_O \delta(1, r) + U_I \delta(2, r)) \right] (m^*)^{(r-5/2)} T^{(r-1/2)}, \quad (22)$$

where U_A , U_O , U_I are the constants, which depend on the nature of the semiconductor crystal and the scattering mode of charge carriers in the crystal – the subscripts A , O , I are for the scattering by the acoustic phonons, the optical phonons, and the ions in the crystal lattice, respectively. $\delta(\dots)$ is the Kronecker symbol.

$I_{ij}(T, \mu^*)$ is the dimensionless functional and is given by:

$$I_{ij}(T, \mu^*) = \int_0^\infty x^i \frac{(x + \beta(T)x^2)^{(rj-j/2+3/2)}}{(1 + 2\beta(T)x)^{2j}} \left(-\frac{df_0}{dx} \right) dx. \quad (23)$$

In formula (23), $\beta(T) = \frac{kT}{E_g}$ is the parameter of nonparabolicity of the energy spectrum

– this parameter is near zero in the semiconductor crystals with a wide energy gap E_g , and the nonparabolic Kane's band turns into a parabolic band, i. e. the semiconductor crystal has a parabolic energy spectrum, $x = \varepsilon / kT$.

We substitute the value $J_{ij}(T, \mu^*)$ from (20) in formulae (1)–(9) of the kinetic properties. For example, the selected kinetic properties of the crystal take the forms:

$$n(T, \mu^*) = N_c(T) I_{00}(T, \mu^*), \quad (24)$$

$$R(T, \mu^*) = \frac{1}{ze n(T, \mu^*)} \frac{I_{00}(T, \mu^*) I_{02}(T, \mu^*)}{I_{01}(T, \mu^*)^2}, \quad (25)$$

$$\alpha(T, \mu^*) = \left(\frac{k}{ze} \right) \left(\frac{I_{11}(T, \mu^*)}{I_{01}(T, \mu^*)} - \mu^* \right), \quad (26)$$

$$N(T, \mu^*) = \left(\frac{k}{e} \right) U_H(T, \mu^*) \left[\frac{I_{11}(T, \mu^*)}{I_{01}(T, \mu^*)} - \frac{I_{12}(T, \mu^*)}{I_{02}(T, \mu^*)} \right], \quad (27)$$

$$\sigma(T, \mu^*) = e n(T, \mu^*) U^{(r)}(T) \frac{I_{01}(T, \mu^*)}{I_{00}(T, \mu^*)}, \quad (28)$$

$$U_D(T, \mu^*) = U^{(r)}(T) \frac{I_{01}(T, \mu^*)}{I_{00}(T, \mu^*)}, \quad (29)$$

$$U_H(T, \mu^*) = |\sigma R| = U^{(r)}(T) \frac{I_{02}(T, \mu^*)}{I_{01}(T, \mu^*)} = U_D(T, \mu^*) \frac{I_{00}(T, \mu^*) I_{02}(T, \mu^*)}{I_{01}(T, \mu^*)}. \quad (30)$$

From formulae (24)–(30) for the kinetic properties of the semiconductor crystals, it will be obvious that in order for the theoretical determination of these properties, the following experimental data should be known: the parameter of scattering r (i. e. the actual scattering mode or modes in the crystal), the reduced chemical potential μ^* , and the carriers' effective mass m^* . There are well-known experimental techniques capable of determining these experimental data [11–13] including the actual scattering modes from the temperature dependence of the Hall mobility U_H .

5. Conclusions

We have proposed general formulae for the kinetic properties of semiconductor crystals expressed in terms of the Fermi integrals.

The formulae constitute the mathematical model of the charge carriers transport phenomena in the semiconductor crystals.

Possibly, it is useful to have a method of evaluating these properties without resorting to the Boltzmann equation, we do then not use this equation as the basis for our model and our analysis of the kinetic properties of semiconductor crystals.

The theoretical analysis of the model shows that for the calculation of the kinetic properties of isotropic doped semiconductor crystals, the following experimental data should be known: the actual scattering mode or modes in the crystal, the reduced chemical potential, and the carriers' effective mass.

References

- [1] Askerov B.M., *Electron transport phenomena in semiconductors*, World Scientific, Singapore 1994.
- [2] Budzak Ya.S., Waclawski T., *Physics of semiconductor crystals. Introduction to statistical theory of thermal and kinetic properties of crystal with computational examples in MATHCAD*, Wydawnictwo Miniatura, Kraków 2008 (in Polish).
- [3] Budzak Ya.S., Vakiv M.M., *Elements of statistical theory of thermal and kinetic properties of semiconductor crystals*, Liga-Pres, Lviv 2010 (in Ukrainian).
- [4] Haug H., Jauho A.-P., *Quantum Kinetics in Transport and Optics of Semiconductors*, Springer 1996.
- [5] Madelung O., *Physics of III-V compounds*, Wiley, New York 1964.
- [6] Polya G., Szego G., *Aufgaben und Lehrsaetze aus der Analysis*, vol. 1, sect. 2, subsec. 1, ex. 77, Springer-Verlag, Berlin-Goettingen-Heidelberg-New York 1964.
- [7] Seeger K., *Semiconductor Physics*, Springer-Verlag, Wien New York 1973.
- [8] Yu P.Y., Cardona M., *Fundamentals of semiconductors. Physics and Materials Properties*, Springer, 2010.
- [9] Zawadzki W., *Mechanisms of Electron Scattering in Semiconductors*, PAN Publishers, Wroclaw-Warszawa-Krakow 1979.
- [10] Budzak Ya.S., *Investigation of Transport Phenomena in Semiconductors with Complex Spectrum*, D.Sc. Thesis, S. Petersburg 1985.
- [11] Pustelny T., Opilski A., Pustelny B., *Determination of Some Kinetic Parameters of Fast Surface States in Silicon Single Crystals by Means of Surface Acoustic Wave Method*, Acta Phys. Polon. A **114**, 2008, 183-190.
- [12] Dziawa P., Kowalski B.J., Dybko K., Buczko R., Szczerbakow A. et al., *Topological crystalline insulator states in $Pb_{1-x}Sn_xSe$* , Nat. Mat. Lett. **11**, 2012.
- [13] Young D.L., Coutts T.J., Kaydanov V.I., Mulligan V.P., *Direct Measurement of Density-of-States Effective Mass and Scattering Parameter in Transparent conducting Oxides Using Second-Order Transport Phenomena*, National Renewable Energy Laboratory, Golden, <http://www.doe.gov/bridge> (online: 09.09.2015).

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STABILISING FEEDBACK IN MAX-PLUS LINEAR MODELS OF DISCRETE PROCESSES

STABILIZACYJNE SPRZĘŻENIE ZWROTNE W MAX-PLUS LINIOWYCH MODELACH PROCESÓW DYSKRETNÝCH

Abstract

This article relates to a synthesising output feedback that is used to control a network of discrete events. The feedback stabilises the system without reducing its initial throughput and its synthesis is mainly based on the theory of residues and the Kleene operator. This article suggests some theoretical results and mathematical foundations of max-plus algebra theory, and in particular, discusses various other aspects of controlling discrete processes and their modelling in the context of a linear max-plus system.

Keywords: discrete processes, TEG, max-plus linear system, stability, feedback control

Streszczenie

Artykuł dotyczy syntezy sprzężenia zwrotnego w sterowaniu siecią zdarzeń dyskretnych. Sprzężenie zwrotne służy do stabilizacji systemu bez zmniejszenia jego początkowej przepustowości i jego synteza opiera się głównie na wynikach teorii residuów i operatora Kleene'a. W artykule zasygnalizowano pewne wyniki teoretyczne i wprowadzono matematyczne podstawy max-plus algebry. Omówiono także inne aspekty sterowania procesami dyskretnymi oraz ich modelowanie w kategoriach liniowego systemu max-plus.

Słowa kluczowe: procesy dyskretne, liniowy system max-plus, stabilność, sterowanie, sprzężenie zwrotne

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1. Introduction

The name dynamic discrete event systems (DDES) applies to a group of systems, usually of a human design, exhibiting a dynamic behaviour. It includes output systems (flexible workshops, assembly lines) [8], communication networks, computer networks [24] and transportation systems (road, rail or air) [28, 34]. It is widely known that to achieve an effective design and operation techniques of different discrete event systems (DES's), methods and tools are needed to model complex material and control flows, to analyse the behaviour and interactions of manufacturing resources and to predict performance measures such as productivity, cycle times and work-in-process.

Many other discrete event dynamic systems, such as transportation networks [35] and communication networks [18], are subject to synchronisation phenomena. DDES often appear in the context of parallel computing, especially in application specific integrated circuits (ASICs) [21], project management systems [19], or telecommunication networks etc. The diversity of these systems naturally leads to different models, including ones based on finite state automata [12], Markov chains [1] and Petri nets [38].

The thesis about the importance of systems in our society leads many researchers to propose mathematical models that describe the behavior of these systems in order to evaluate their performance, optimization, control and design. Recent years have seen a quantitative growth of research on discrete systems that can be modelled as max-plus linear systems (MPLSs). Previously, the majority of literature on this class of system discussed the modelling, performance and analyses of their properties [2, 20] rather than their control.

A Petri net model playing the role of an event graph, max-plus algebra and min-plus algebra make it possible to write linear equations and model the aforementioned systems. Timed event graphs (TEG's) form a subclass of timed Petri nets and are suitable for modelling these systems. A TEG is a timed Petri net in which all places have exactly one upstream transition and one downstream transition. Its description can be transformed into a (max; +) or a (min; +) linear model and vice versa [2].

A single transition for each output of a place has the practical result that all possible conflicts concerning the use of tokens in the places have already been established. This means that there is no competition on either side of the transition but this is the unavoidable cost of maintaining the linearity of the system. Some problems must be accepted for these restrictions to be guaranteed.

More than two decades have now passed since the max-plus algebra and similar algebraic tools started to play a central role in the simulation and analysis of discrete processes. Actually, however, the use of these theories led to the creation of a linear theory for certain discrete event systems [16]. Therefore, working in the field of linear systems, max-plus uses guidelines and concepts presented by classical theories. However, the number of researchers involved in this new area of DES system theory is rather small compared to the hundreds of other scientists who contributed to the classical theory. Historically, some specific control theory for DDESs was developed including, for example, optimal feedback [26, 31, 37], open loop control [27, 29, 32] and predictive control for a perturbed system [36].

To handle more complex models, it was necessary to adapt mathematical tools while keeping most of the concepts provided by earlier developments. Differential geometry,

power series of noncommutative variables and differential algebra have all been used to develop such models for which essential questions concerning, for example, controllability and observability, stabilisation and feedback synthesis, etc., have been revisited. Max-plus, min-plus and other idempotent semiring structures turn out to be the right mathematical tools to at best, bring linearity back, or to at least achieve a certain suitability to the nature of the phenomena to be described in this field of DES.

This paper presents the problem of stabilising discrete processes of the system by using output feedback control. The issue of stabilising DES was considered by Commault [9] and Cohen [8]. Just as in classical models of continuous systems, stability is closely linked to the structure of the system. A TEG is structurally stable if its number of markers is restricted for all sequences of data input [2].

Commault defines a sufficient condition for the stability of the TEG which is satisfied if the network graph strongly associates events by closing the circuit from the exit to the entry. Some quantity of markers which limit the marking of the system should be added along the extra path (in the simplest method).

Furthermore, it is shown in [2] that a controllable and observable TEG can be stabilised by adding feedback output without changing its own bandwidth. Gaubert showed in [15] that the number of tokens that must be placed in the feedback to stabilize the TEG represents the optimisation of resources that can be formulated as an integer linear programming problem.

The approach presented here is based on Gaubert's work [15]. Other methods are presented as the greatest linear feedback in [30], where this represents the synthesis of dynamic feedback which minimizes the number of markers required to retain the original throughput.

Section 2 serves as a reminder of the algebraic tools necessary for synthesising feedback. Section 3 briefly explains max-plus modelling with a practical example. Section 4 shows how the existing feedback in a TEG can improve the stability of the system and how it can be implemented. Section 5 aims to present an illustrative example.

2. The mathematical basis of max-plus algebra

This section aims to concisely present formal definitions and key algebraic tools used in subsequent sections. More precisely, it reiterates Chapter 4 of [2] and [7] and certain theses [10, 17, 25, 30]. The max-plus theory is based on the lattice theory and largely reverses applications defined on ordered sets [4, 11]. Algebra max-plus use a structure of idempotent semiring algebras S .

This semiring S has two internal operations denoted by \oplus and \otimes which have many properties:

- operation \oplus is associative, commutative and idempotent, that is $a \oplus a = a$;
- operation \otimes is associative (but not necessarily commutative) and distributive on the left and on the right with respect to the \oplus ;
- neutral elements for \oplus and \otimes are represented by ε and e respectively;
- ε is an absorbing element of the right $\forall a = \varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$;

- just like in classical algebra, operator \otimes will often be omitted in equations (as in classical multiplication) and $a^i = a \otimes a^{i-1}$ as well as $a^0 = e$.

$Z_{\max} = \{Z, -\infty, \infty\}$ is the set of numbers endowed with the maximisation as the operator \oplus (symbol pronounced ‘o-plus’) and the sum as the \otimes operator (pronounced ‘o-times’) and neutral values $\varepsilon = -\infty$ and $e = 0$.

For example:

$$y = 14 \otimes 3^2 \oplus 3 \otimes 5^8 = \max((14 + 2 \times 3), 3 + 8 \times 5) = 43$$

The implicit equation $x = a \otimes x \oplus b$ determines $a = a^* \otimes b$ where the Kleene star operator:

$$a^* = \bigoplus_{i=0}^{\infty} a^i \quad (1)$$

For matrices if $\mathbf{A}, \mathbf{B} \in Z_{\max}^{m \times n}$ and $\mathbf{C} \in Z_{\max}^{m \times p}$, then:

$$(\mathbf{A} \oplus \mathbf{B})_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij})$$

$$(\mathbf{A} \otimes \mathbf{C})_{ij} = \bigoplus_{k=1}^n a_{ik} \otimes c_{kj} = \max_{k=1 \dots n} (a_{ik} \otimes c_{kj}) \text{ for all } i, j.$$

The Kleene star operator can also be applied to matrices:

$$\mathbf{A}^* = \bigoplus_{i=0}^{\infty} \mathbf{A}^i \quad \text{with} \quad \mathbf{A}^{i+1} = \mathbf{A} \otimes \mathbf{A}^i \quad \text{and} \quad \mathbf{A}^0 = \mathbf{I} \quad (2)$$

where the identity matrix \mathbf{I} that is $(\mathbf{I})_{ij} = \begin{cases} e & \text{if } i = j \\ \varepsilon & \text{if } i \neq j \end{cases}$

Equation (2) which has a nilpotent matrix, achieves convergence (all coefficients ε).

3. Timed Event Graph as Max-Plus Linear System

A Petri net is called an event graph if all arcs have the weight 1 and each place has exactly one input and one output transition, that is, $|^*p_i| = |p_i^*| = 1, \forall p_i \in P$. P is the (finite) set of places but $|^*p_i|, |p_i^*|$ is the quantity of the input and output transition, respectively of, place p_i . In general (standard) Petri nets, and consequently also event graphs, only the logical behaviour and the precise ordering of the events, that is, the possible sequences of firings of transitions, are modelled. However, in many applications, the timing of the events plays an essential role and specific firing times or the earliest possible firing times of transitions are of particular interest. Consequently, event graphs have been equipped with timing information. Time can be associated either with transitions (representing transition delays) or with places (representing holding times) and provides the TEG.

In TEGs, transition delays can always be converted into holding times (by simply shifting each transition delay to all input places of the corresponding transition). However,

it is not possible to convert every TEG with holding times into a TEG with transition delays. In a TEG with holding times, a token entering place has to spend some time units before it can contribute to the firing of its output transition. The example of a TEG with the determined holding time of 2 units in place P1 is given in Fig. 1 [5].

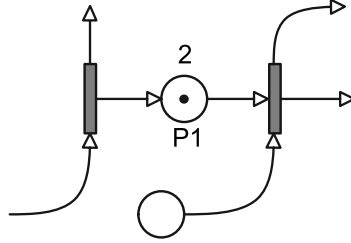


Fig. 1. Graphical representation of a TEG

State-space descriptions in the max-plus algebra for a certain class of discrete-event-systems become linear representations which are similar to state-space equations in the traditional modern control theory [36]. Generally speaking, for any TEG system, one obtains the following kind of equations as a MPLS [6]:

$$\mathbf{x}(k) = \bigoplus_{i=0}^M \mathbf{A}_i \mathbf{x}(k-i) \oplus \mathbf{B}_i \mathbf{u}(k-i) \quad (3.1)$$

$$\mathbf{y}(k) = \bigoplus_{i=0}^M \mathbf{C}_i \mathbf{x}(k-i) \quad (3.2)$$

where \mathbf{x} , \mathbf{u} , and \mathbf{y} are vectors of dimensions equal to the numbers of internal, input and output transitions, respectively. \mathbf{A}_i , \mathbf{B}_i , and \mathbf{C}_i are matrices of the appropriate dimensions with entries in the max-plus algebra, and M is the maximal number of tokens in the initial marking. The variables of (3) are time instances and the represented events occur at k -times. Entries of \mathbf{A}_0 correspond to places with no tokens and (3) has an implicit form.

The implicit part can be eliminated by successive substitutions or by using the Kleene star (2). Finally, the MPLS has an explicit form:

$$\mathbf{x}(k) = \mathbf{A} \mathbf{x}(k-1) \oplus \mathbf{B} \mathbf{u}(k) \quad (4.1)$$

$$\mathbf{y}(k) = \mathbf{C} \mathbf{x}(k) \quad (4.2)$$

where:

$\mathbf{A} \in \mathbb{Z}_{\max}^{m \times n}$ – state-transition (or system) matrix,

$\mathbf{B} \in \mathbb{Z}_{\max}^{m \times p}$ – input matrix,

$\mathbf{C} \in \mathbb{Z}_{\max}^{m \times n}$ – output matrix,

$\mathbf{x} \in \mathbb{Z}_{\max}^n$ – state vector,

$\mathbf{u} \in Z_{\max}^p$ – input (control) vector,

$\mathbf{y} \in Z_{\max}^m$ – output vector.

As there are transforms similar to the Laplace and Z-transforms (used to represent continuous and discrete time trajectories in the classical theory), the input-output behaviour of a TEG can be represented by another transfer relation.

For trajectories $x(k)$, this can be the γ -transform $X(\gamma) = \bigoplus_{k \in \mathbb{Z}} x(k) \gamma^k$ where γ can be considered as the backward shift operator i.e. $\gamma x(k) = x(k-1)$.

Below, the model of the system is represented by (γ, δ) – transform – $M_{in}^{ax}[[\gamma, \delta]]$ a set of formal power series for two variables γ and δ . A finite series of $M_{in}^{ax}[[\gamma, \delta]]$ is a polynomial and is used to code a set of information concerning the transition of a TEG. Monomial $\gamma^k \delta^t$ may be interpreted as the k -th event occurring at least at time t .

4. Synthesis of a feedback system

The feedback stabilisation requires a certain quantity of starting markers in the feedback, e.g. when the TEG describes a production system with markers for the initial resource, which may be any transport or recycling equipment (robots, pallets or raw materials). For computational processes, extra resources can represent some units of memory. Consequently, it is of particular importance to limit their quantity as far as possible. What is being considered is the problem of minimising the marking for both restrictions (stabilise and preserve throughput). The problem of optimising resources is solved by the $(\min, +)$ algebra in [14] and also by other authors who solved the problem, possibly without defining an approach $(\max, +)$. The TEG system consists of m inputs and p outputs, as well as some arcs of places between outputs and inputs to make the system strongly consistent.

The problem is solved using an the algebraic structure $M_{in}^{ax}[[\gamma, \delta]]$ of the open system

$$\mathbf{y} = \mathbf{H} \mathbf{u} \quad (5)$$

where:

$$\mathbf{H} \in M_{in}^{ax}[[\gamma, \delta]]_{p \times m}.$$

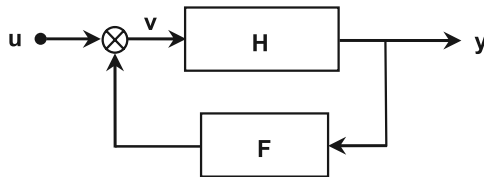


Fig. 2. Block diagram of a system with feedback

By applying (2), the closed system (Fig. 2) is

$$\mathbf{y} = \mathbf{H}(\mathbf{F}\mathbf{H})^* \mathbf{u} \quad (6)$$

where:

$\mathbf{F} \in M_{in}^{ax}[[\gamma, \delta]]_{p \times m}$ – output feedback

The feedback system is a matrix transfer (transmittance) with elements $F_{ij} = \gamma^{q_{ij}}$ where arcs containing the q_{ij} markers connect the output y_j with the input signal u_i at the initial state. If $F_{ij} = \epsilon$, then there is no arc. In practice, it is not always necessary to connect all outputs to all inputs for a strong relationship.

The stabilisation problem refers to calculating and minimizing $q = \{q_{ij}\}$ in a closed-loop system to maintain the same throughput as an open loop has. Authors of [14] have shown that this problem can be solved as an integer linear programming task with a linear cost function

$$J(q) = \sum_{i=1}^{i=m} \sum_{j=1}^{j=p} \alpha_{ij} q_{ij} \quad (7)$$

where:

α_{ij} – weighting factor associated with the price of each resource to maintain the same throughput with a restrictive condition

$$\lambda(q) \geq \lambda_p \quad (8)$$

where:

λ_p – throughput for the open loop system,

$\lambda(q)$ – throughput for the feedback system.

Throughput $\lambda(q)$ is derived from the formula

$$\lambda(q) = \min_c \frac{w_{Nc}(q)}{w_{Tc}} \quad (9)$$

where:

$w_{Nc}(q)$ – sum of markers in circuit c ,

$w_{Tc}(q)$ – sum of holding times in circuit c .

From (8) and (9), the following constraints are satisfied for each cycle:

$$w_{Nc}(q) \geq \lambda_p w_{Tc} \quad (10)$$

To summarise, the stabilisation method of solving the problem with limited resources consist of the following steps:

- determine the transform of \mathbf{H} for an open system (5);
- formulate transform feedback \mathbf{F} (6);
- solve the problem of integer programming (7, 9) and get the values of \mathbf{F} ;
- verify the results (e.g., by a simulation).

Feedback \mathbf{F} stability provides a closed system retaining the same throughput and minimising the cost function.

5. Example

An example of the practical application of the above described theoretical considerations is presented here. Consider the controllable and structurally observable marked TEG in Fig 3. Its matrix transfer as $M_{in}^{ax}[[\gamma, \delta]]$ is obtained as follows:

$$\mathbf{H} = \begin{bmatrix} \delta^{10} + \gamma^2 \delta^{12} (\gamma \delta)^* & \delta^8 (\gamma^2 \delta^4)^* & \delta^5 (\gamma^2 \delta^3)^* \\ \varepsilon & (\delta^{11} + \gamma \delta^{12}) (\gamma^2 \delta^4)^* & \varepsilon \end{bmatrix} \quad (11)$$

This TEG represents a system process consisting of 5 units P1 to P5. Due to the difference in the throughput of the units, it is noted that the model is not a stable because is accumulated markers (T2, P53) before of the P3 (Fig 4). The stability of this system can be obtained by adding a feedback output and this is enough to make the TEG strongly consistent and achieve stability. In this particular case, it becomes highly coherent by adding TEG feedback in the form of:

$$\mathbf{F} = \begin{bmatrix} \gamma^{q_{11}} & \varepsilon \\ \gamma^{q_{21}} & \gamma^{q_{22}} \\ \gamma^{q_{31}} & \varepsilon \end{bmatrix} \quad (12)$$

The problem here can be considered as the optimisation of resources in order to minimise the following cost function (9)

$$\lambda(q) = \min \left(\frac{q_{11}}{10}, \frac{q_{21}}{8}, \frac{q_{22}}{11}, \frac{q_{31}}{5} \right) \quad (13)$$

This problem can be solved by taking into account the sum of expression tokens and times for each elementary circuit determining the rate of the TEG process.

A naive algorithm for calculating the elementary circuits is simpler than writing a linear program which is unpractical for large graphs i.e. for full graph vertices, the complexity is $O((n - i)!)$. Gaubert's approach [12] allows only the n^2 inequality to be considered.

The feedback \mathbf{F}_R (12) which stabilises the system while maintaining its original performance and minimising the amount of resources (marked in Fig. 3 as q_{ij} places) is:

$$\mathbf{F} = \begin{bmatrix} \gamma^5 & \varepsilon \\ \gamma^4 & \gamma^6 \\ \gamma^3 & \varepsilon \end{bmatrix}$$

Subsequent figures present some results for the considered example (interpolated to make them more visual): time evolution for $u_1 = u_2 = u_3 = \mu$ in an open system Fig. 4 and closed system Fig. 5 and Fig. 6.

Results have been calculated using a self-developed software package based on

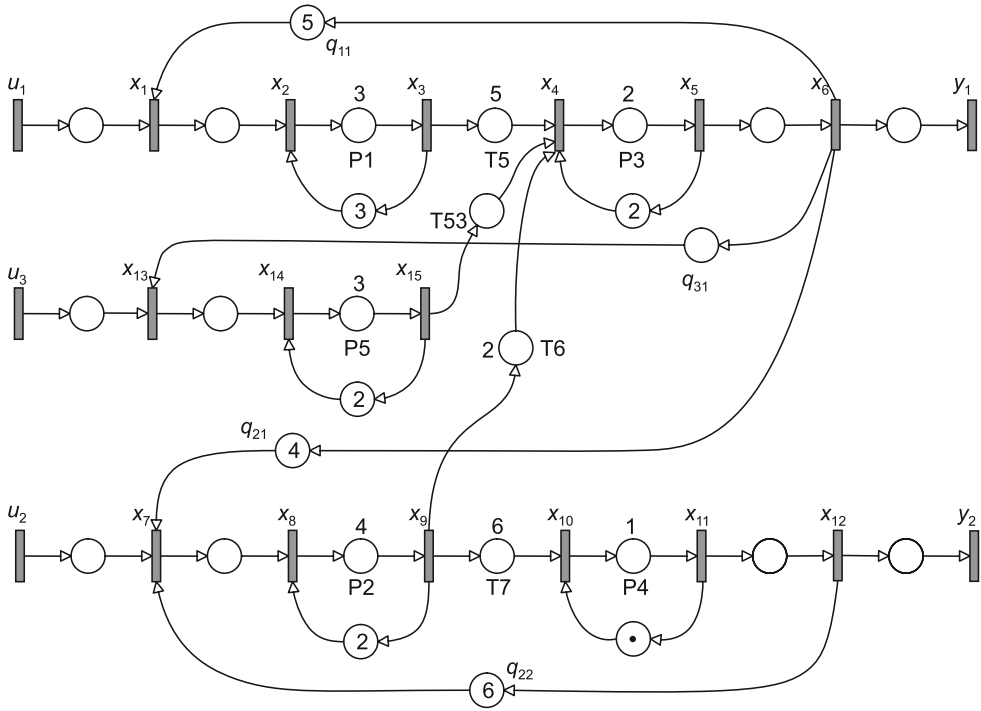


Fig. 3. TEG of the system with feedback

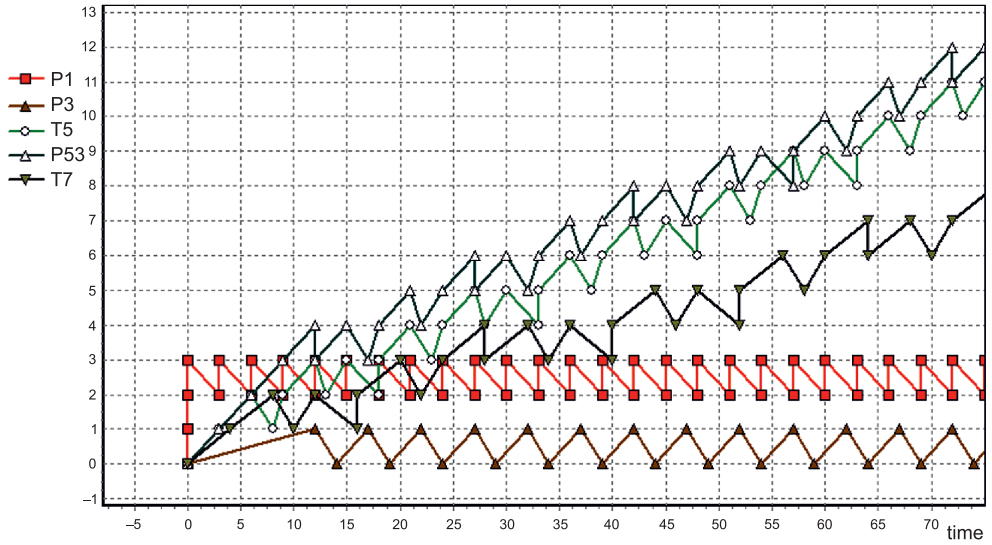


Fig. 4. Time evolution states of markers in open system

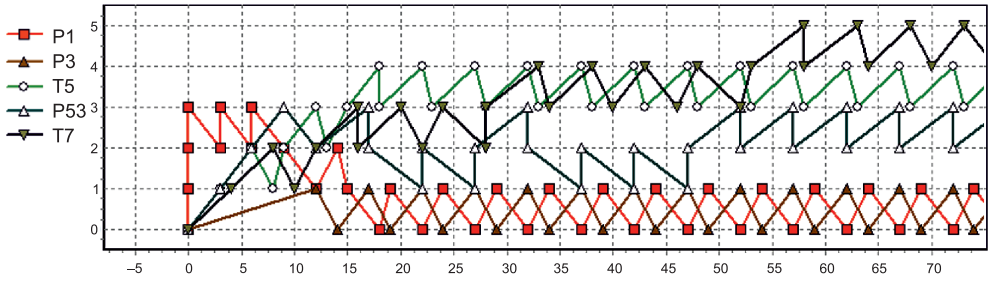


Fig. 5. Time evolution states of markers in closed system

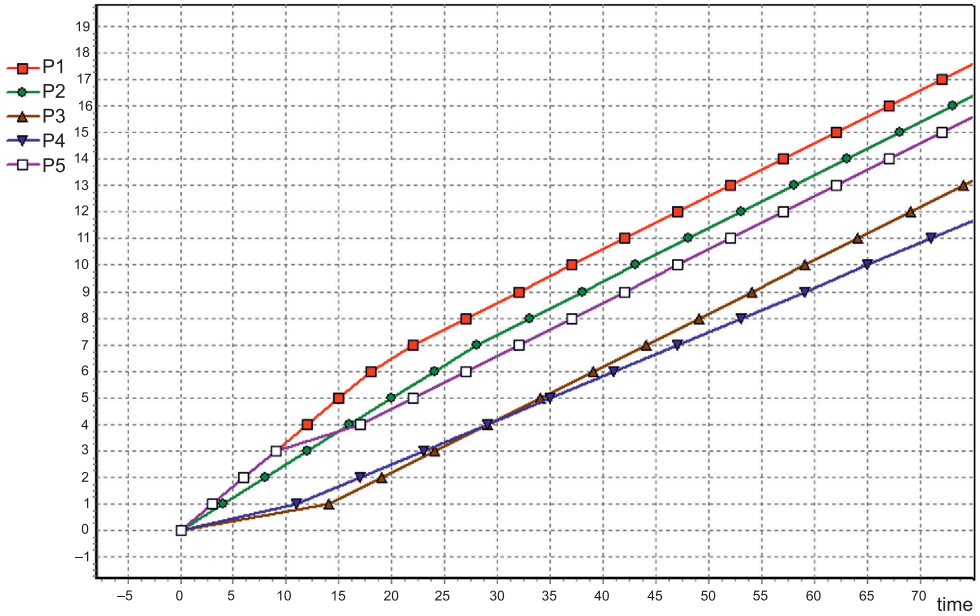


Fig. 6. Time evolution count of activity processes in closed system

Borland libraries TeeChart, Component and Software Tools for Manipulating Periodic Series: <http://www.istia.univ-angers.fr/hardouin/outils.html> and Toolbox TINA <http://projects.laas.fr/tina/home.php>.

6. Conclusions

This paper presents the synthesis of stabilisation feedback control for limited resources. It describes the synthesis of feedback control to stabilise the quantity of resources. This is only an initial example of the application of the max-plus systems theory and further research will include applying the system to larger practical system optimisation procedures

and synthesising process control with state feedback as well as using models for predictive and adaptive control. More than a decade ago, the authors [6] concluded that: “By comparison with classical linear system theory, there are areas which are practically untouched, mostly because the corresponding mathematical tools are yet to be fabricated”. Nowadays, problems of using max-plus systems are still a reality [3, 13, 15, 22, 23, 25, 33].

References

- [1] Applebaum D., *Markov processes, semigroups and generators*, Cambridge Studies in Advanced Mathematics, No. 93, p. 120-189.
- [2] Bacceli F., Cohen G., Olsder G., Quadrat J., *Synchronization and Linearity. An Algebra for Discrete Event Systems*, John Wiley & Sons Ltd, 1992.
- [3] Bezem M., Nieuwenhuis R., *Hard problems in max algebra, control theory, hypergraphs and other areas*, Information Processing Letters, 110(4), 2010, p. 133-138.
- [4] Blyth T., Janowitz M., *Residuation Theory*, Pergamon press 1972.
- [5] Brunsch T., Hardouin L., Raisch J., *Formal Methods in Manufacturing: Modelling Manufacturing Systems in a Dioid Framework*, Campos J. ed., Part 1, Ch. 2 2, p. 29-94.
- [6] Cohen G., Gaubert S., Quadrat J., *Max-plus algebra and system theory: Where we are and where to go now*, Annual Rev. Elsevier-IFAC, 1999, Vol. 23, No. 1, p. 207-219.
- [7] Cohen G., *Residuation and applications. Algèbres Max-Plus et applications en informatique et automatique*, Ecole de printemps d’informatique théorique, 1998.
- [8] Cohen G., Dubois D., Quadrat J., Viot M., *A linear system theoretic view of discrete event processes and its use for performance evaluation in manufacturing*, IEEE Trans. on Automatic Control, 1985, AC-30, p. 210-220.
- [9] Commault C., *Feedback stabilization of some event graph models*, IEEE Transaction. on Automatic Control, 43(10), October 1998, p. 1419-1423.
- [10] Cottenceau B., Hardouin L., Boimond J.-L., Ferrier J.-L., *Synthesis of greatest linear feedback for Timed Event Graphs in dioid*, IEEE Transactions on Automatic Control, 1999, Vol. 44, No. 6, p. 1258-1262.
- [11] Davey B., Priestley H., *Introduction to Lattices and Order*, Cambridge University Press 1990.
- [12] D’Souza D. Shankar P., *Modern Applications of Automata Theory*, World Scientific Publishing Co Pte Ltd, 2012.
- [13] El Hichami O., *An algebraic method for analysing control flow of BPMN models*, International Journal of Applied Mathematics & Computer Science (in revision 2015).
- [14] Gaubert S., *Resource Optimization and (min, +)*, IEEE Transactions on Spectral Theory Automatic Control, Vol. 40, Issue 11, 1995, p. 1931-1934.
- [15] Gaubert S., *Theorie des systemes lineaires dans les dioides. These de doctorat*, Ecole des Mines de Paris, Paris 1992.
- [16] Hardouin L., *Sur la Commande des Systèmes (max,+) Linéaires*, DEA Automatique et Informatique Appliquée – Angers, Décembre 2004.
- [17] Hardouin L., Menguy E., Boimond J.-L., Ferrier J.-L., *SISO Discrete Event Systems Control in Dioids Algebra*, Special issue of Journal Européen des Systèmes Automatisés (JESA), Vol. 31, No. 3, 1997, p. 433-452.
- [18] Heidergott, B., Olsder, G.J., van der Woude, J., *Max Plus at Work, Modeling and Analysis of Synchronized Systems*, Princeton University Press, 2006.
- [19] Jamroz L., Raszka J., *Applying methods supporting project management information system process*, Automatyzacja Procesów Dyskretnych, Vol. I, Gliwice 2010, p. 57-64.

- [20] Jamroz L., Raszka J., *Simulation method for the performance evaluation of system of discrete cyclic processes*, Proceedings of the 16-th IASTED International Conference on Modelling, Identification and Control, Innsbruck, Austria, 17–19.02.97, p. 190-193.
- [21] John M., Smith S., *Application-Specific Integrated Circuits*, Addison-Wesley Publishing Company, VLSI Design Series, June 1997.
- [22] Karatkevich A., Wisniewski R., *Polynomial Algorithm for Finding a State Machine Cover of Petri Nets*, Applied Mathematics & Computer Science (in revision 2015).
- [23] Katz R., Schneider H., Sergeev, S., *Commuting matrices in max-algebra* (Preprint 2010/03). University of Birmingham, School of Mathematics, 2010.
- [24] LeBoudec J.-Y., Thiran P., *Network Calculus*, Springer Verlag 2002.
- [25] Lhommeau M., *Etude de systèmes à événements discrets dans l'algèbre (max,+): 1. Synthèse de correcteurs robustes dans un dioïde d'intervalles. 2. Synthèse de correcteurs en présence de perturbations*, Thèse, LISA – Université d'Angers, 2003.
- [26] Lhommeau M., Hardouin L., Cottenceau B., *Optimal control for (max,+) – linear systems in the presence of disturbances. Positive Systems: Theory and Applications*, POSTA, Springer LNCIS 294, 2003, 47-54.
- [27] Lhommeau M., Hardouin L., Ferrier J.-L., Ouerghi I., *Interval analysis in dioid: Application to robust open loop control for timed event graphs. Decision and Control*, 2005, European Control Conference, CDC-ECC'05, 44th IEEE, p. 7744-7749.
- [28] Lotito P., Mancinelli E., Quadrat J.P., *A minplus derivation of the fundamental car-traffic law*, Report 324, INRIA 2001.
- [29] Maia C.A., Hardouin L., Santos-Mendes R., *On the Model Reference Control for Max-Plus Linear Systems*, In 44th IEEE CDC-ECC'05 Sevilla, 2005, p. 7799-7803.
- [30] Maia C., *Identification et Commande de systèmes à événements discrets dans l'algèbre (max,+)*, Thèse, LISA – Université d'Angers, Université de Campinas – Brésil 2003.
- [31] Maia C.A., Hardouin L., *Optimal closed-loop control of Timed Event Graphs in Diod*, IEEE Transactions on Automatic Control, 48 (12), 2003, p. 2284-2287.
- [32] Menguy E., Boimond J.-L., Hardouin L., Ferrier J.-L., *Just in time control of timed event graphs: update of reference input, presence of uncontrollable input*, IEEE Transactions on Automatic Control, 45(11), 2000, p. 2155-2159.
- [33] Merlet G., *Semigroup of matrices acting on the max-plus projective space*, Elsevier, Linear Algebra and Its Applications, 432, 2010, p. 1923-1935.
- [34] Subiono O.G., Gettrick, M.M., *Course notes: On large scale max-plus algebra model in railway systems. Algèbres Max-Plus et applications en informatique et automatique*, Ecole de printemps d'informatique théorique.
- [35] Farhi Q.N., Goursat M., *Derivation of the fundamental traffic diagram for two circular roads and a crossing using minplus algebra and Petri net modeling*, Decision and Control, 2005, European Control Conference, CDC-ECC'05, 44th IEEE.
- [36] Schutter B.D., van den Boom T., *Model predictive control for max-plus-linear discrete event systems*, Automatica, 37(7), 2001.
- [37] Shang Ying, Sain M.K., *On zero semimodules of systems over semirings with applications to queueing systems*, [in:] American Control Confer., 2005, p. 225-230.
- [38] Szpyrka M., *Sieci Petriego w modelowaniu i analizie systemów współbieżnych*, WNT, Warszawa 2008.

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SYMBOLIC TENSOR CALCULUS — FUNCTIONAL AND DYNAMIC APPROACH

ZASTOSOWANIA PROGRAMOWANIA FUNKCYJNEGO I DYNAMICZNEGO DO SYMBOLICZNEGO RACHUNKU TENSOROWEGO

Abstract

In this paper, we briefly discuss the dynamic and functional approach to computer symbolic tensor analysis. The *ccgrg* package for Wolfram Language/Mathematica is used to illustrate this approach. Some examples of applications are attached.

Keywords: computer algebra, symbolic tensor calculus, functional programming, dynamic programming, Mathematica, Wolfram Language

Streszczenie

Krótko omawiamy zastosowania programowania dynamicznego i funkcyjnego do symbolicznego rachunku tensorowego. Demonstrując ten rodzaj programowania, posługujemy się pakietem *ccgrg.m* napisanym dla języka Wolfram Language/Mathematica. Zamieszczamy kilka przykładów ilustrujących funkcjonalność pakietu.

Słowa kluczowe: algebra komputerowa, symboliczny rachunek tensorowy, programowanie funkcyjne, programowanie dynamiczne, Mathematica, Wolfram Language

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1. Introduction

Cartesian tensors are identified with matrices. In contrast, in curved coordinates, even more so in a curved space, viewing the tensor matrix *in extenso* is not instructive. The subject of interpretation are equations, invariants, symmetries, and observables, while the tensor components values — except for certain privileged frames of reference — are of minor importance. This follows directly from the general principle of covariance, which states that the physical sense is independent of the choice of coordinate system, whereby each coordinate choice is allowed.

The vast majority of the computer tensor tools — including those build in Wolfram Language/Mathematica, the symbolic language which we use here — identify tensors with matrices, providing them with procedures for tensor contraction, tensor product and the rules of symmetry. Objects of such a class require the evaluation of complete tensor matrices. A sequence of intermediate steps evaluates quantities, most of which are irrelevant to the actual goal of the calculation, and are an unnecessary burden. A few tensor packages can operate on abstract level, where the differential manifold is not specified. They cover the entire tensor algebra, and essentially, form the sophisticated programming languages [8, 12, 13]. On the other hand, all of them depend on the primary computer algebra languages they employ (Mathematica, Maple) — the languages which constantly develop. The risk of losing compatibility between the primary language and the computer algebra package is greater the more complex the package.

The project we realized in 2013–2014 was aimed at constructing an open source package [1] for Mathematica, which defines some basic concepts and rules, and is open to development by users. The syntax is kept as close as possible to standard textbook notation [2] and typical scheme of manual calculations. On the other hand, we employ dynamic programming with modern computational techniques, a *call-by-need strategy* and *memoization*. The question is not how much can be computed, but how flexible the software can be? The results are discussed in this paper.

We have taken into account that physicists have in mind co- and contravariant tensor components, rather than individual tensors of a precised valence (covariant or contravariant tensors). Raising and lowering tensor indices is a basic manual technique which comes directly from the Einstein convention. What physicist actually do is they aggregate tensors of arbitrary valence (all combinations of upper and lower indices) into a single object. The choice of particular valence is left for the future, and decided upon later. In order to stick to that, while the matrix representation is employed, one would need to evaluate and store all the co- and contravariant index combinations. In the case of the Riemann tensor in four dimensions, this gives 8^4 expressions, while only a small proportion of them will probably be used further on. Although they are not independent as subjects to tensor symmetries, the benefits of this fact are not straightforward. Modern techniques of differentiation, and reducing terms are quick enough, while the evaluation of conditions *If* still cost time. As a result, computing the same component twice may cost only slightly more than retrieving it from another matrix element. But there still remains the problem of memory. In the matrix approach, thousands of array's components are build of long algebraic expressions.

The functional representation of tensors allows one to avoid some of these problems. In this paper, we focus on the *ccgrg*-package (the open source code written for *Mathematica*

9–10, and distributed from Wolfram Library [3]). The goal of the article is to argue for some particular computation technique. We do not give here the precise tutorial to the package. The same approach to tensors in Python is presented in [4].

2. Dynamic paradigm of programming

The idea is not new, as it dates back to the famous Richard Bellman book *Dynamic Programming* (1957). Bellman introduces the concept of the goal function. This means that the starting point is identical to the final expression in the search. To evaluate this expression, the algorithm automatically splits tasks into sub tasks down to elementary level. The tree of calls is not explicitly known to programmers, and the computational process does not require any intelligent interference. For tensors, this means that the subroutines do not evaluate all the components but some of them, acting on demands of routines from a higher level. Most tensor components remains unknown. Only these are computed which contribute to the final expression. This process is known as *lazy evaluation* or *call-by-need strategy* [5, 6]. To evaluate Ricci scalar for Schwarzschild metrics (33), *ccgrg* calls 16 Riemann components. 40 Riemann components are needed for full Ricci tensor, and 808 Riemann components for Carminati–McLenaghan \mathcal{W}_1 invariant [7] $\mathcal{W}_1 = \frac{1}{8}(C_{abcd} + \mathbf{i} * C_{abcd})C^{abcd}$ build of the Weyl C and dual Weyl $*C$ contractions. In each case, the appropriate components are selected automatically and the rest of components remains unevaluated.

Tensors form a class of functions, where the tensor indices are the innermost group of arguments. For instance, the covariant derivative we denote as:

$$T_{ij;m} \rightarrow \nabla[T][i, j, m], \quad (1)$$

$$T_{ijk;mn} \rightarrow \nabla[T][i, j, k, m, n]. \quad (2)$$

The outer argument contains the tensor name, while the inner argument lists the tensor indices. In the case of Lie derivative

$$\mathcal{L}_U V_i \rightarrow \text{LieD}[U][V][i], \quad (3)$$

$$\mathcal{L}_U T_{ij} \rightarrow \text{LieD}[U][T][i, j], \quad (4)$$

the outermost argument U is the vector field that generates the dragging flow, the middle (V or T) is the tensor name, and the innermost, according to the general rule, specify indices. The grouping of arguments, and the appropriate ordering of groups allows one to work with headers $\nabla[T]$, $\text{LieD}[U][T]$, etc., treating them as operators. Inserting new groups of outer arguments does not affect the methods governing tensor index operations. Indexes run through $\{1, \dots, \text{dim}\}$ for covariant components and through $\{-\text{dim}, \dots, -1\}$ for contravariant components (the inverse Parker–Christensen notation [8]). Thus we write:

$$T_{ij} \rightarrow T[i, j], \quad (5)$$

$$T_i^j \rightarrow T[i, -j], \quad (6)$$

$$T^{ij} \rightarrow T[-i, -j], \quad (7)$$

$$T^{ij}_k \rightarrow T[-i, -j, k], \quad (8)$$

and so on. Tensor valence operations (conversions from covariant to contravariant components, and vice versa), and the covariant differentiation are the only methods of the object. Other tensor operations like contractions, products, etc. are realized by elementary algebra and the Einstein convention. Conversions are called just by placing or removing the sign “minus” in front of selected tensor indices.

3. The manifold

Tensors live on a manifold. This manifold is declared in the opening procedure `open[x, g]`, where the list of coordinate names x and metric tensor g are the arguments. When the manifold is ‘opened’, tensors are evaluated on this particular manifold. The metric tensor can be uniquely defined, or may contain arbitrary functions of coordinates. In the last case, `open` indicates the class of manifolds distinguished by the specified structure of the metric tensor.

Tensors are created by setting object data in the form of the pair: tensor matrix, and the associated tensor valence (all covariant components as a default). By calling `tensorExt[matrix, valence]` one creates a tensor object which can return components for arbitrary valence, on demand. Tensors which are in frequent use (Ricci, Riemann, Weyl, the first and the second fundamental forms on hypersurfaces), as well as covariant and Lie derivatives are predefined in the *ccgrg*-package. Tensor components are not evaluated in advance, only when called for the first time. This is guaranteed by the `SetDelayed` (`:=`) command. For instance, when calling the Riemann curvature tensor in the Schwarzschild space-time one obtains:

$$\begin{aligned} R_{1212} \rightarrow \text{In}[1] &:= \text{tRiemann}[1, 2, 1, 2] \\ \text{Out}[1] &:= -\frac{2M}{r^3} \end{aligned} \quad (9)$$

$$\begin{aligned} R_{121}^2 \rightarrow \text{In}[2] &:= \text{tRiemann}[1, 2, 1, -2] \\ \text{Out}[2] &:= -\frac{2M(2M-r)}{r^4} \end{aligned} \quad (10)$$

$$\begin{aligned} R_{ijmn}R^{ijmn} \rightarrow \text{In}[3] &:= \text{Sum}[\text{tRiemannR}[i, j, m, n]\text{tRiemannR}[-i, -j, -m, -n] \\ &\quad \{i, \text{dim}\}, \{j, \text{dim}\}, \{m, \text{dim}\}, \{n, \text{dim}\}] \\ \text{Out}[3] &:= 48\frac{M^2}{r^6} \end{aligned} \quad (11)$$

The tensor rank is fixed, except for the covariant derivative, where each differentiation rises the rank by one. Covariant differentiation is realized by header ∇ (or its full name, `covariantD`), and by extending the list of indices by one. Extension of this list by more than one returns derivatives of a higher order.

$$\begin{aligned} R_{121}^2{}_{;2} \rightarrow \text{In}[1] &:= \nabla[\text{tRiemannR}][1, 2, 1, -2, 2] \\ \text{Out}[1] &:= -\frac{6M(2M-r)}{r^5} \\ R_{121}^2{}_{;22} \rightarrow \text{In}[2] &:= \nabla[\text{tRiemannR}][1, 2, 1, -2, 2, 2] \end{aligned} \quad (12)$$

$$\text{Out}[2] := -\frac{6M(9M^2 - 4Mr)}{r^6} \quad (13)$$

Rising the tensor rank by differentiation does not affect the general ability to rise or lower indices by choosing appropriate signs, both for the original and the differentiation indices. The flexibility of tensor indexes stems directly from the functional representation of tensor and the *call-by-need strategy*.

4. Memoization and symmetries

Trees of calls for different tasks may have nonempty intersections (and typically this is the case). The `SetDelayed` function which enables *call-by-need strategy* at the same time forces the multiple evaluation of components whenever trees of calls overlap. To avoid unwilling effects, one needs to store computed values, and to have a mechanism that can recognize known and unknown components. This process is called *memoization*. The term *memoization* was introduced by Donald Michie in 1968 [9] and refers to the class of functions that can learn what they found in past calls. In *C* or Python, the *memoization* tools are usually constructed by programmers (see [4]). Wolfram Mathematica provides dedicated tool in the core language. The construction consists in the recursive definition of the form

$$f[x_]:=f[x]=\text{expr}[x,\dots] \quad (14)$$

where the same name of the function appears twice, first followed by `SetDelayed` and next, by `Set`. The definition of f which is initially equivalent to $\text{expr}[x,\dots]$, is successfully supplemented by the values found during each function call. The memoized (memorized) values form cache. The cache is searched first; therefore, no tensor component is evaluated twice. This means that each of the expressions in the examples (10)–(14) is evaluated only once. Whenever called again, the values are instantly retrieved from the cache. The same refers to expressions called by subroutines. In the Schwarzschild space time (33), the Carminati–McLenaghan invariants \mathcal{W}_1 and \mathcal{W}_2 , when evaluated separately, call 808 and 552 Riemann components, respectively. Jointly, they call 1046 Riemann components which means that 296 components are taken from the cache.

Calls appeal to tensor symmetries. The symmetry rules are the argument conditions within the same memoizing function scheme

$$\mathfrak{f}[x_]; \langle \text{test}[x] \rangle := \mathfrak{f}[x] = \text{expr}[x,\dots] \quad (15)$$

For instance, for the Ricci tensor R_{ij} , the conditional definition takes the form

$$\begin{aligned} \text{tRicciR}[i_ , j_]; i > j &:= \text{tRicciR}[i, j] = \text{tRicciR}[j, i] \\ \text{tRicciR}[i_ , j_] &:= \text{tRicciR}[i, j] = \text{Sum}[\text{tRiemannR}[-s, i, s, j], \{s, \text{dim}\}] \end{aligned} \quad (16)$$

with positive i, j (covariant components). The rules of symmetry are introduced directly in the definition of the tensor. In order to recognize Ricci symmetries, the kernel does not refer to the symmetries of the Riemann tensor. Subsequently, the Riemann tensor symmetries are not the result of the evaluation of Christoffel symbols, etc. Symmetries defined as the argument conditions in lazy-evaluated functions put constraints directly on computational processes.

5. The confidence of results in dynamic programming

Dynamic programming allows one to reach the goal at a minimal cost of time and memory. The *call-by-need strategy* avoids computing unnecessary components. Memoization protects against multiple evaluations. The cost we pay, however, is confidence in the results. Functions that can learn and remember are reluctant to forget. The user must not change the tensor definition during the same session. The cache has priority over evaluation. A new definition does not affect the values which are already found. If definitions change, one may readily collect a mixture of results belonging to different definitions.

The content of the cache together with the general tensor definition can be viewed by the Mathematica command `Definition[tensorname]`. Based on this command, `cacheview[tensorname]` returns the list of the *memorized* tensor components. For instance

$$\begin{aligned} \text{In}[26] &:= \text{cacheview}[\text{tRiemannR}] \\ \text{Out}[26] &:= \{\{\text{tRiemannR}, \{-4, 1, 4, 1\}\}, \{\text{tRiemannR}, \{-3, 1, 3, 1\}\}\} \end{aligned} \quad (17)$$

means that R^4_{141} and R^3_{131} Riemann components have been already called and stored in the cache. To view all the memoized quantities which refer to `tRiemannR`, one writes

$$\begin{aligned} \text{In}[27] &:= \text{associated}[\text{tRiemannR}] \\ \text{Out}[27] &:= \{\{\{\text{tRiemannR}, \{-4, 1, 4, 1\}\}, \{\text{tRiemannR}, \{-3, 1, 3, 1\}\}\}, \\ &\quad \{\text{covariantD}[\text{tRiemannR}], \{1, 2, 1, 2, 2\}\}\} \end{aligned} \quad (18)$$

Commands

$$\begin{aligned} \text{In}[31] &:= \text{retreat}[\text{tRiemannR}] \\ \text{In}[32] &:= \text{retreat}[\text{tRiemannR}, \text{associated}] \end{aligned} \quad (19)$$

clear the stored values displayed in (17) and (18), respectively. In contrast to the core-language *Mathematica* commands `Clear` and `Unset`, the command `retreat` does not affect the general definition of a tensor. Calling `cacheview` after `retreat` returns the empty list.

For sake of safety, the name of each memoized tensor shall be appended to `memRegistry`. This is realized by `erasable[tensorname]`. `memRegistry` allows the automatic erasing of the whole cache content whenever `open[x, g]` is called. User may safely redefine the manifold with no risk of unwanted remnants from the past computations. *ccgrg* is equipped with extensive random tests of memory clearance. The above mentioned security tools do not prevent some incidental mistakes such as those shown below

$$\begin{aligned} \text{In}[41] &:= f[k_]:=f[k]=k; f/@\text{Range}[8]; \\ \text{In}[42] &:= f[k_]:=f[k]=1/k; f/@\text{Range}[8]; \end{aligned}$$

To effectively implement *memoization* to algorithms, minimal training and discipline are indispensable.

6. Tensor definition scheme

This is the place to show the typical construction of a tensor. We choose the induced metrics (the first fundamental form) $h_{ij} = g_{ij} - v_i v_j$ on the hypersurface orthogonal to the vector field v_i in the space with the metric g_{ij} .

```
erasable/@{hcov,h};
hcov[v_-][j_-,i_-]/;i<j:=hcov[v][j,i]=hcov[v][i,j];
hcov[v_-][i_-,j_-]:=g[i,j]-v[i]v[j]/vectorsquared[v]//simp[];
h[v_-][i_-,j_-]/;index[i,j]:=h[v][i,j]=tensorExt[hcov[v]][i,j]//simp[];
```

(20)

In the first line we append `memRegistry`. The next line contains the symmetry settings. The third line defines the matrix of the covariant tensor (`hcov` is not a tensor in the meaning of the *ccgrg* package and will be invisible in the general context.) The last line creates the tensor `h` as an object containing data (`hcov`), methods to control the index values (condition `index`), and methods to rise or lower indices (provided by the `tensorExt` procedure).

Following the scheme above, users may define their own tensors and append to the package, or just use them in notebooks. Summation, multiplying tensors by numbers or functions (not containing indices as arguments!), and covariant differentiation return tensors which not need to be separately defined as in (20). Yet, even in these cases, the definition scheme (20) is strongly recommended to assure effectiveness.

7. Examples of use

7.1. Differential operators in an arbitrary coordinate system

In technical sciences there is a need to express differential operators in some particular curved coordinates or on curved surfaces. The task may appear as a part of the heat transport or diffusion problems for systems of more complex geometry. Many typical gradient or Laplacian expressions are catalogued in the literature, but still the range of geometries that can be encountered in nature is much richer. Below, we choose a catenoid — one of the best known minimal surfaces — for which, we hope, Laplacian is not published in print.

The family of catenoids numbered by r and parametrized by u and v define a curved coordinate system in the Euclidean space. The first step is to find the Euclidean metric form in coordinates r, u, v (the core Wolfram Language)

```
In[1]:= Needs["ccgrg"];
In[2]:= x[r_-,u_-,v_-]=rCosh[v/r]Cos[u];
In[3]:= y[r_-,u_-,v_-]=rCosh[v/r]Sin[u];
In[4]:= z[r_-,u_-,v_-]=v;
In[5]:= crd={r,u,v};
In[6]:= form=Dt[x[r,u,v]]^2+Dt[y[r,u,v]]^2+Dt[z[r,u,v]]^2//FullSimplify
```

$$\begin{aligned} \text{Out}[6] := & \text{Cosh}\left[\frac{v}{r}\right]^2 (r^2 \text{Dt}[u]^2 + \text{Dt}[v]^2) + \frac{\text{Dt}[r]^2 (r \text{Cosh}\left[\frac{v}{r}\right] - v \text{Sinh}\left[\frac{v}{r}\right])^2}{r^2} + \\ & \frac{\text{Dt}[r] \text{Dt}[v] (v - v \text{Cosh}\left[\frac{2v}{r}\right] + r \text{Sinh}\left[\frac{2v}{r}\right])}{r} \end{aligned} \quad (21)$$

The second step is to evaluate the covariant differentiation $f_{,i}{}^i$ of an arbitrary function f on the manifold with the metric form (21):

$$\begin{aligned} \text{In}[7] &:= \text{open}[\text{crd}, \text{toMatrix}[\text{form}, \text{crd}]]; \\ \text{In}[8] &:= \text{Sum}[\nabla[f[r, u, v]][i, -i], \{i, \text{dim}\}] // \text{FullSimplify} \\ \text{Out}[8] &:= \frac{r^2 f^{(2,0,0)}[r, u, v]}{(r - v \text{Tanh}\left[\frac{v}{r}\right])^2} + \frac{\text{Sech}^2\left[\frac{v}{r}\right] f^{(0,2,0)}[r, u, v]}{r^2} + f^{(0,0,2)}[r, u, v] + \\ & \frac{2 r f^{(1,0,1)}[r, u, v]}{v - r \text{Coth}\left[\frac{v}{r}\right]} - \frac{(r \text{Sinh}\left[\frac{2v}{r}\right] + 2v)^2 \text{Sech}^4\left[\frac{v}{r}\right] f^{(1,0,0)}[r, u, v]}{4 (r - v \text{Tanh}\left[\frac{v}{r}\right])^3} \end{aligned} \quad (22)$$

The only time-consuming operations are simplifications `FullSimplify`, which are inevitable in this code. Similar computations can be easily performed for an arbitrary transformation defined by inputs the `In[2]–In[4]`.

7.2. Riemann curvature invariants: the case of spherical symmetry

Carminati–McLenaghan invariants [7] form a complete set of Riemann invariants for the spacetime which obey the Einstein equations with the hydrodynamic energy-momentum tensor. These invariants read

$$\mathcal{R}_1 = \frac{1}{4} S^a{}_b S^b{}_a, \quad (23)$$

$$\mathcal{R}_2 = -\frac{1}{8} S^a{}_b S^b{}_c S^c{}_a, \quad (24)$$

$$\mathcal{R}_3 = \frac{1}{16} S^a{}_b S^b{}_c S^c{}_d S^d{}_a, \quad (25)$$

$$\mathcal{W}_1 = \frac{1}{8} (C_{abcd} + i^* C_{abcd}) C^{abcd}, \quad (26)$$

$$\mathcal{W}_2 = -\frac{1}{16} (C_{ab}{}^{cd} + i^* C_{ef}{}^{ab}) C_{cd}{}^{ef} C_{ef}{}^{ab}, \quad (27)$$

$$\mathcal{M}_1 = \frac{1}{8} S^{ad} S^{bc} (-i^* C_{abcd} + C_{abcd}), \quad (28)$$

$$\mathcal{M}_2 = \frac{1}{8} i^* C_{abcd} S^{bc} S_{ef} C^{aefd} + \frac{1}{16} S^{bc} S_{ef} (C_{abcd} C^{aefd} - i^* C_{abcd} i^* C^{aefd}), \quad (29)$$

$$\mathcal{M}_3 = \frac{1}{16} S^{bc} S_{ef} C_{abcd} C^{aefd} + \frac{1}{16} S^{bc} S_{ef} i^* C_{abcd} i^* C^{aefd}, \quad (30)$$

$$\mathcal{M}_4 = -\frac{1}{32} S^{ag} S^c{}_d S^{ef} C_{ac}{}^{db} C_{befg} - \frac{1}{32} S^{ag} S^c{}_d S^{ef} i^* C_{ac}{}^{db} i^* C_{befg}, \quad (31)$$

$$\mathcal{M}_5 = \frac{1}{32} S^{bc} S^{ef} (i^* C^{aghd} + C^{aghd}) (*C_{abcd} *C_{gefh} + C_{abcd} C_{gefh}). \quad (32)$$

S_{ab} stands for the Plebanski tensor (the traceless Ricci tensor $S_{ab} = R_{ab} - \frac{1}{4}Rg_{ab}$). Invariants \mathcal{W}_1 and \mathcal{W}_2 are complexes, which are defined by means of contractions of the Weyl C and the dual Weyl $*C$ tensors.

In the case of vacuum and spherically symmetric spacetime (Black Hole)

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (33)$$

the complete computation appears as follows:

```

In[1] := Needs["ccgrg"];
In[2] := x = {t, r, ϑ, ϕ};
In[3] := g := DiagonalMatrix[-(1 - 2M/r), (1 - 2M/r)^-1, r^2, r^2 Sin[ϑ]]^2];
In[4] := $Assumptions = And@@{0 < t, 0 < r, 0 < ϑ < π, 0 < ϕ < 2π, 0 < M};
In[5] := open[x, g];
In[6] := {CMinvR1, CMinvR2, CMinvR3}
Out[6] := {0, 0, 0}
In[7] := {CMinvW1, CMinvW2}
Out[7] := {6M^2/r^6, -6M^2/r^6}
In[8] := {CMinvM1, CMinvM2, CMinvM3, CMinvM4, CMinvM5}
Out[8] := {0, 0, 0, 0, 0}
In[9] := {TimeUsed[], $MachineType, $Version, $ProcessorType,
          $ProcessorCount}
Out[9] := {8.80264, PC, 10.0forLinuxx86(64-bit)(December4, 2014), x86-64, 2}

```

(34)

As see result of these operations, two nonvanishing invariants were found, while 1320 of 4096 components of the Riemann tensor object were evaluated and memorized. For the PC computer with AMD64 processor and Linux Mint x86 (64-bit), the whole operation took 8.80264 seconds CPU.

8. Final remarks

The degree of complexity in a symbolic programming is much less predictable than in numerical computation. This is due to the fact that the quantity of algebraic terms in an expression is not a uniquely determined number. This number significantly depends on the ability of the algorithm to recognize mathematical identities, and for the same function, may differ in orders of magnitude. In this circumstance, the economic style of the dynamic programming (*call-by-need strategy*, *memoization*) may take the first rank role.

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References

- [1] Woszczyna A., Kycia R. A., Golda Z. A., *Functional Programming in Symbolic Tensor Analysis*, Computer Algebra Systems in Teaching and Research, Vol. IV, No 1, pp. 100–106, 2013.
- [2] Synge J. L., Schild A., *Tensor Calculus*, University of Toronto Press, 1959.
- [3] Woszczyna A., et al.: *ccgrg — The symbolic tensor analysis package, with tools for general relativity*, 2014, <http://library.wolfram.com/infocenter/MathSource/8848/>.
- [4] Czaja W., *GraviPy*, <https://pypi.python.org/pypi/GraviPy/0.1.0>.
- [5] Hudak P., *Conception, Evolution, and Application of Functional Programming Languages*, ACM Computing Surveys **21**, pp. 383–385, 1989.
- [6] Reynolds J. C., *Theories of programming languages*, Cambridge University Press, 1998.
- [7] Carminati J., McLenaghan R. G., *Algebraic invariants of the Riemann tensor in a four-dimensional Lorentzian space*, J. Math. Phys. **32**, 3135, 1991.
- [8] Parker L., Christensen S. M., *MathTensor: A System for Doing Tensor Analysis by Computer*, Reading, MA: Addison-Wesley, 1994.
- [9] Michie D., *Memo Functions and Machine Learning*, Nature, No. 218, 1968, pp. 19–22.
- [10] Wald R. M., *General Relativity*, The University of Chicago Press, 1984.
- [11] [Tensorial homepage] <http://home.comcast.net/~djmpark/TensorialPage.html>.
- [12] [GRTensor homepage] <http://grtensor.phy.queensu.ca/>.
- [13] [xAct homepage] <http://www.xact.es/>.
- [14] Korol'kova A. V., Kulyabov D. S., Sevastyanov L. A., *Tensor computations in computer algebra systems*, Programming and Computer Software, 39 (3), 2013, pp. 135–142.

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EMERGENCE OF COOPERATION AS A RESULT OF MUTATION AND INHERITANCE IN PD/PG-LIKE GAME

WSPÓŁPRACA JAKO REZULTAT MUTACJI I DZIEDZICZENIA W GRACH TYPU PD/PG

Abstract

The approach of evolutionary games like prisoner's dilemma, among others, is based on a consistent strategy. We propose an experimentally verified system where cooperation can arise out of two simple factors: mutation and inheritance. This system has a social dilemma property and allows each agent to set its own desired number of participants. There are two essential mechanisms: fluctuation through which mutation leads to cooperation, and natural selection which tends to promote cheaters and therefore disrupts the cooperation. It is shown in numerical simulations that the interplay between both mechanisms leads to an equilibrium and that no intentional strategies are necessary to establish and sustain cooperation. Thus, starting from a population of non-cooperating agents, natural evolution can end with a population composed of cooperating groups with the mean group size determined by the fluctuation rate and the pay-off function. A thorough analytical explanation of numerical results is provided.

Keywords: social dilemma, cooperation, prisoner's dilemma, evolutionary games, public goods game

Streszczenie

Podejście w grach ewolucyjnych, takich jak Dylemat więźnia, jest oparte przede wszystkim na spójnej strategii. W prezentowanym artykule zaproponowane zostało podejście, w którym współpraca dwóch więźniów może wynikać z mutacji oraz dziedziczenia. System ten ma charakter dylematu społecznego, w którym możliwe jest ustawienie przez każdego z agentów pożądanej dla niego liczby uczestników gry. Ponadto zostały zdefiniowane dwa podstawowe mechanizmy: wahania, które w wyniku mutacji prowadzi do współpracy, a także doboru naturalnego, który ma tendencję do promowania oszustów, w związku z czym zakłóca proces współpracy. W symulacjach numerycznych zostało wykazane, że wzajemne oddziaływanie między dwoma mechanizmami prowadzi do równowagi, a niecelowe strategie są konieczne do nawiązania i utrzymania współpracy między agentami. Tak więc, zaczynając od populacji nie współpracujących agentów, w wyniku procesu ewolucji, może skończyć się na populacji złożonej ze współpracujących grup ze średniej wielkości grupami określonymi przez stopień wahań oraz funkcję spłacającą. Artykuł dokładnie wyjaśnia wyniki numerycznych testów.

Słowa kluczowe: dylemat społeczny, współpraca, dylemat więźnia, gry ewolucyjne, gra dobra publicznego

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1. Introduction

Cooperation is fundamental to numerous biological and social systems. Multicellular organisms, social insects, herds of animals and human societies are all examples of systems dependent on cooperative interactions. Explaining the emergence of cooperation has been a challenge since the time of Darwin. Cooperators are defined as those who help others at a cost to themselves, while the term ‘defectors’ is used to describe those who receive the benefits of altruism without providing any help in return.

With regard to cooperation, the players are usually given the options of working alone, trusting another in cooperation, or cheating another who has trusted them. The choice is usually done by a consistent strategy (an algorithm). Certain pieces of research demonstrate ‘wars’ of played by actors utilising different strategies. The aim of finding the best strategy, i.e. the one which defeats the others, is achieved through evolutionary selection.

Cheating requires individuals who are being cheated; therefore, not all actors can enter into a cheating strategy at any one time. Too many cheaters also undermine the fraction of cooperators that may lead to the extinction of the cooperators. An equilibrium is therefore desirable, where a specified fraction of cooperators is able to displace another fraction of cheaters.

In contrast to concepts where the cooperation is a result of consistent strategy exercised in a game of skill, our system shows that the cooperation can be a product of two simple factors: mutation and inheritance, both of which are widely observed in nature. For instance, micro-organisms are not expected to make rational decisions. Consequently, an important question immediately arises of whether or not it is possible to propose a social dilemma model capable of evolving towards cooperation when driven only by these two basic natural phenomena. Discovering such models is as important for our understanding of the emergence of cooperation, as models based on rational strategies. In the present paper, we propose a social dilemma game which extends the public goods game [1] and leads to cooperation supported by mutation and inheritance only.

In the public goods game [1], the players contribute to a common pool which is then enhanced by a linear function and distributed evenly among the contributors. Our system investigates a comparable approach, with the main difference being that the enhancement is provided by a power function. With specific restrictions over the exponent value, we are able to introduce social dilemma to this problem. Thus, by contributing, an agent exposes themselves to a potential risk of being cheated, over the prospect of receiving profitable return. The question regarding such an arrangement is how much the agents become willing to contribute.

Another major difference is that there is no common pool in our system. Each agent establishes their own pool with a number of agents from their neighbourhood. The outcome of participation is shared evenly. The problem of reciprocity arises – if an agent A has included B, will B return the favour? By the fact that the agent A is in neighbourhood of a few other agents, another agent C, as a replacement for B, may include A in its group, that will make A treated fairly. This may enable a society-wide indirect reciprocity where each agent is cheated by another but cheats another themselves. Whether such an arrangement would work and under which conditions is another question that is answered in this paper. A certain

biological analogy here is a multicellular organism where a single unit (a cell, an organ, a system) provides a service for other units whilst simultaneously receiving services of other units, not necessarily the same units to which they are providing a service.

2. Related Works

The emergence of cooperation has been studied based on tools developed within the framework of evolutionary games theory [2] to emphasise the problem of cooperation and defection for the evolution of social behaviour [3]. The prisoner's dilemma [4] game is a model game used to study aspects related to corruption [5], incorrect social information [6], shared rewards [7], multiple players [8], fraternity and egoism [9]. Similarly, the snowdrift game [10] has been used to study the problems of structured populations [11], random rematching [12], invasion and expansion of cooperators [13], and other problems. In the stag-hunt [14] game, individual fitness and past group decisions are taken into consideration [15]. The problems of shared rewards [7], sanctions as honest signals [16] and the optimal sizes of groups [4] have been studied within the framework of the volunteer's Dilemma game [17]. Future research should be directed into the area of automatic crawling systems [18], where cooperation can improve crawling performance. The cooperation modelling could be based on Petri net modelling [19] so that the formal and informal presentation could be clarified. The public goods game seems to be a basis for tender behaviour; therefore, results of such research could have a practical application [20, 21]. Additionally, research in the area of emotions and reactions [22] should be performed because they play an equally essential role as rational reasoning when it comes to judging the intelligence of a partner. Another important aspect of the research field is the species formation process [24] in agent-based algorithms, where the basis of this process is sexual selection and the persistent genetic diversity of the population [23].

3. Aim of the study

The aim of the study was to investigate the effect of the mutation and the process of succession to evolutionary games such as the prisoner's dilemma game. In the present method, a higher degree of freedom was introduced which allows the player to determine the number of participants in the game. In this system, the process of natural selection tends to favour cheats, while the second main mechanism, fluctuation, leads to cooperation as a result of mutation. As mentioned, natural selection favours cheats, this means that this process disrupts cooperation.

It is shown at a later stage of testing the system, that the interaction between these two mechanisms leads to balance, and the mutation is an integral factor in maintaining the already established cooperation between agents.

Consequently, even if the initial population consists of non-cooperating groups of agents, as a result of an evolutionary process, it may become a population that consists of the cooperating group of agents.

4. Methodology

In this section, we propose methodology which is experimentally verified in section 4. There are M agents in the system. During the game, each agent can establish cooperation with its neighbours. For simplicity of analytical derivation, we assume that:

1. the number of neighbours is fixed for all agents and equal to system parameter N
2. reciprocity: if A is a neighbour of B , then B is a neighbour of A

We assume that the game is played in ring topology and each agent has a neighbourhood of $\frac{N}{2}$ agents in both directions. Our numerical simulations show that populations defined over random networks behave identically to ring topology, on average, only if N is small enough to keep the above assumptions correct.

The lifetime of an agent is divided into the following steps:

1. Within the neighbourhood, establish a group with other agents
2. Receive a profit from participation in own and other groups
3. Produce an offspring if possible

The establishment of a group of size $0 \leq k \leq N$ is governed by the agent's genetic trait p and population-wide fluctuation rate q . The genetic trait $p = \frac{k}{N}$, with a willingness to cooperate, is established in the agent's direct predecessor as fraction of the size of its group. It should be expected that an agent bearing such a trait establishes a group of the same size as its predecessor; however, we want the fluctuation to interfere in this process. The decision to include a particular neighbour is made by a stochastic process accommodating both p and q in following formula:

$$B(p) \text{ xor } B(q) \tag{1}$$

repeated N times for each one *agent* in the neighbourhood, where $B(x)$ stands for a single Bernoulli trial, returning *true* with the probability of x . The exclusive-or (*xor*) is the only logical operator that provides deserved symmetry – it allows fluctuation to affect the genetic trait both ways: if $B(p)$ produces a negative decision, it still can be changed with the probability of q ; additionally, if $B(p)$ gives a positive decision, it may be revoked with the same probability.

Once the groups are established, the participants are awarded for their participation. For each agent, the total income is represented as $W_A + W_C$, where W_A is the share from a group the agent has established, and W_C is the sum of the shares from other groups where the agent was included. With the trait p , the agent has an influence over the part W_A but has no control over W_C as this depends on other agents and their p . The cooperation emerging from the game proposed in this paper regards this exact interdependence of the agents. The next section will discuss in detail the problem of fair or unfair cooperation.

The last step in an agent's lifetime is reproduction. Each agent presents its genetic trait and income. The genetic trait $p = \frac{k}{N}$ is different from the one that the agent receives from its predecessor due to a stochastic process and an interference from fluctuation q (1). A new

generation of M agents is created. For each new agent, its trait p is taken from an agent of the departing generation, chosen by the means of geometric probability where the income is the weight.

The games of dilemma are typically characterised by four parameters: income as a traitor T , income in fair cooperation R , income in no cooperation P and income when being cheated S . A simplified example of only two players A and B will be helpful: let the income function per player be $\frac{\sqrt{n}}{n}$, where n is the number of participants in the given group. If both A and B work alone, then each earns $P = 1$. If A makes B a participant, then both of them earn $\frac{\sqrt{2}}{2} < 1$, but B has the option to reciprocate. If so, then both will end the round with $R = \sqrt{2} = 1.41$. Otherwise, B cheats A by not returning the favour, then B earns $T = 1 + \frac{\sqrt{2}}{2} = 2.41$ and A earns only $S = \frac{\sqrt{2}}{2} = 0.71$. As shown, this example has the configuration of $T > R > P > S$, exactly like the prisoner's dilemma. It means that this problem is a social dilemma: by going into cooperation, A may improve its income at the risk of losing considerably if B cheats.

In the actual system, we generalise profit function as n^α for $0 < \alpha < 1$. In this interval of α , the property of social dilemma is retained. An agent X assuming a group of size k will receive its share:

$$W_A = \frac{k^\alpha}{k} \quad (2)$$

In order to receive fair return, the agent X must be included in other groups set by agents who have X in their neighbourhood, such that the sum of shares W_C would be equal to:

$$W_R = k^\alpha - \frac{k^\alpha}{k} \quad (3)$$

If $W_C < W_R$, then we can consider the agent X to be cheated, otherwise, X is a cheat. However, even if $W_C < W_R$, it can be such that $W_A + W_C > 1$; therefore, it is not guaranteed that always $P > S$. Certainly, $T > R > P$ for $k > 1$. It is important that in such a system, it is no longer an absolute statement if the cooperation is fair or not. It is relative to the wager, established by the agent setting a group of size k . This shows how different this system is compared to known games of dilemma. The iterated prisoner's dilemma investigates cooperation among n players, but participation is always in pairs. Stag-hunt and snowdrift are games that allow n -player participation, but the number is arbitrary. In the system presented herein, players independently choose the size of the participation pool.

The above considerations are correct for $0 < \alpha < 1$. In case of $\alpha \geq 1$, the participation is disproportionally profitable (the larger the group, the better the income); therefore, agents

trying to gain advantage by cheating are facing a different dilemma to the risk described above. For this reason, the problem in $\alpha \geq 1$ is not in the scope of this paper.

As shown, the outcome of the system is dependent on the values of p which are being carried over the generations. An agent with a lower p relative to the other agents should accumulate greater income than the others. This will result in greater offspring count bearing that low value of p , but cheating in the next generation will be tougher due to a lower average value of p . We expect this dependency to provide a point of equilibrium between cheaters and altruists.

5. Experiment

As already stated, the process of creating a group of participants out of the neighbourhood is stochastic and is governed by two variables p and q , where p is the agent's property inherited from the direct predecessor, and q is a population-wide variable. The result of this process (1) is a set of participants, whose count k will determine the agent's direct share W_A (2), the fair sum of indirect shares W_R (3) and ultimately, the new value of p dubbed

$p' = \frac{k}{N}$ that will be passed to the next generation provided that the agent will be successful

in reproduction. Because the value of p is key in the behaviour of this system, we will now try to estimate its value. First, we will calculate the expected value of formula (1), which we shall call a vertex:

$$\text{vertex}(p, q) = B(p) \text{ xor } B(q) = p \cdot (1 - q) + (1 - p) \cdot q = p + q - 2pq \quad (4)$$

The value of this function has two functions. Firstly, it is the expected value of probability that one particular agent will be added to the group of a given agent A . Secondly, as the formula (1) is executed N times by the agent A in a stochastically independent way, it determines p' . This is the case because the expected value of $k = \text{vertex}(p, q) \cdot N$ and

$p' = \frac{k}{N}$; therefore, $p' = \text{vertex}(p, q)$.

However, the vertex function covers only part of the lifetime of agent. As mentioned before, the evolutionary process shapes the population of agents in a manner according to their income. This process we will call genetix: $(p_1 \dots p_M) = \text{genetix}(p_1 \dots p_M; \alpha)$ which brings us to the full model of the process:

$$(p_1 \dots p_M) = \text{vertex}(\text{genetix}(p_1 \dots p_M; \alpha), q) \quad (5)$$

The above equation treats agents independently – this is only important if their $p - s$ differ due to the stochastic process. In the analytical approach, we will focus on the average value of p that characterises the entire population simultaneously. For the average value, we will reduce (5) to the scalar equation:

$$p = \text{vertex}(\text{genetix}(p, \alpha), q) \quad (6)$$

We will start with an analysis of the vertex function:

$$p = \text{vertex}(p, q) = p + q - 2pq \quad (7)$$

This equation has a fixed point of $p = 0.05$. For $0 < q < 0.5$, the equation has monotonic convergence towards the fixed point. For $0.5 < q < 1$, the equation has oscillating convergence towards 0.5 which introduces a certain level of instability in the process (5). For this reason, this paper will not regard $q > 0.5$ and as q represents mutation, it is unlikely that any real-world evolutionary process would rely on such a high level of mutation.

In order to analyse how the genetix function affects the value of p with the influence of α , we must perform the following steps:

1. Produce the Poisson distribution of agents with their values of p .
2. Assign the expected value of income for each class of distribution.
3. Transform the distribution in a way that an evolutionary process would do.
4. Recalculate the new value of average p out of the transformed distribution.

The first step is achieved as following $P_0 \dots_N$ is a vector in which the k -th coordinate represents the probability that an agent has k cooperators.

$$P_k = \binom{N}{k} \cdot p^k \cdot (1-p)^{N-k} \quad (8)$$

In the second step, we will calculate the expected value of income weighted by frequency P_k . In $R_0 \dots_N$, the k -th coordinate represents the fraction of the whole income pool that is in possession of the k class. It is important to note that out of k , we are only able to calculate the agent's income part W_A . In this manner, it is impossible to calculate the W_C part, as it requires information not available in such plain partitioning. If the population is well-mixed, then all agents should receive a similar amount of W_C . In such cases, this analysis is correct. In the numerical data, it is shown that this accuracy depends on certain circumstances, giving us the ability to determine whether or not the population is well-mixed at a given time.

$$R_k = P_k \cdot \frac{(k+1)^\alpha}{k+1} \quad (9)$$

From the third step, we will receive the distribution as it should be reshaped by the evolutionary process. The R_k are frequencies weighted with respective incomes. $S_0 \dots_N$ normalises those values to 1, which means that the k -th component bears the probability that

a new agent will receive the genetic trait $p = \frac{k}{N}$.

$$S_k = \frac{R_k}{\sum_{j=0}^N R_j} \quad (10)$$

In the last step, we will recalculate the new average p by weighting $\frac{k}{N}$ components of S .

$$p' = \sum_{k=0}^N \frac{S_k \cdot k}{N} \quad (11)$$

p' is value of the scalar version of the genetix for arguments of p and α . Fig. 1 presents Poisson distributions regarding the genetix function with $N = 10$. The initial plot, labelled $p = 0.5$, is derived from equation (8). Further plots, labelled with different values of α are derived from (10) after transformations with respective values of α . As it should be expected, the transformation causes the plots to lean left from the original plot, representing a tendency of going towards $k = 0$. For $0 < \alpha < 1$, the equation $p = \text{genetix}(p, \alpha)$ tends monotonically towards $p = 0.0$. The tendency becomes stronger with a lowering value of α – this means that the genetic selection is a process that disrupts cooperation.

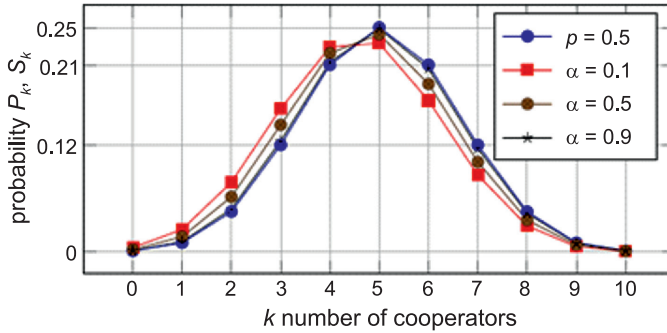


Fig. 1. Different values of α deform the initial distribution of $p = 0.5$

Keeping in mind equation (6) as the simplest model of our system, we learn that its behaviour is an equilibrium of two opposite factors: $\text{vertex}(p, q)$ tends to bring cooperation to the level of $p = 0.5$, while $\text{genetix}(p, \alpha)$ tends to disrupt cooperation down to $p = 0.0$. For a given q and α , Fig. 2 presents the average value of p , calculated from 200 iterations

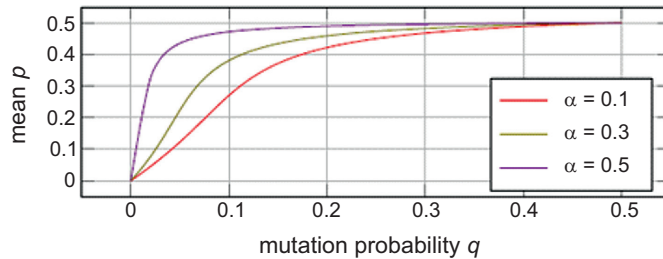


Fig. 2. Analytically calculated mean p for different values of q and α

of (6). Each iteration represents one generation in the system. The plots validate the previous analysis: the higher the q , the stronger p tends to 0.5, and the lower the α , the stronger p tends to 0.0.

We will now investigate the role of mutation in sustaining the cooperation. For fixed values of q and α , we run 200 iterations of (6) where in the first 100 iterations, q is set at a certain level and then changed to zero – this effectively reduces formula (1) to only $B(p)$. With $q = 0.0$, vertex function (7) becomes an identity function. By doing so, we aim to discover if the trait p will sustain the cooperation on its own. As should be expected from the behaviour of the genetix function, the answer is negative. Without fluctuation, there is no mutation and the willingness to cooperate disappears. Fig. 3 and Fig. 4 present the course of this analytical experiment for initial values $q = 0.25$ and $q = 0.05$ respectively. As for the dynamical properties, the higher the rate of decline, the lower is the value of α – this ensures that it is the property of genetic selection that disrupts the cooperation.

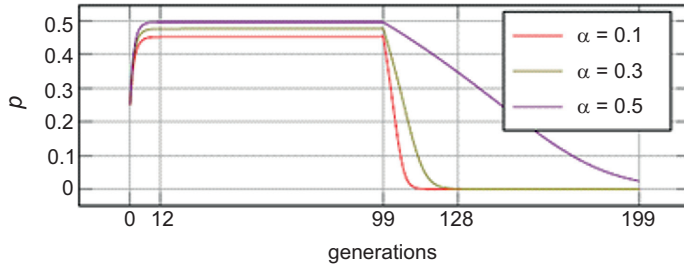


Fig. 3. Analytical solution for $q = 0.25$ in 100 generations and $q = 0.0$

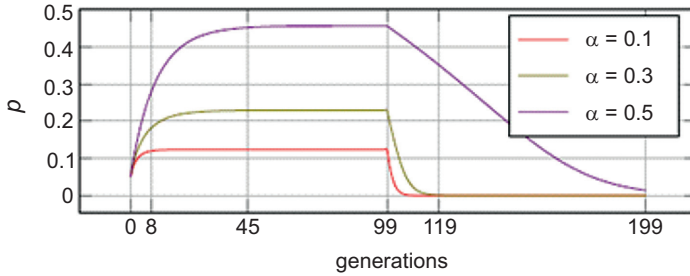


Fig. 4. Analytical solution for $q = 0.05$ in 100 generations and $q = 0.0$

6. Results

In this section, we are presenting the results from running the actual system, as described in section 3. It appears that in a steady state, the system behaves quite like is expected from theoretical considerations with vertex and genetix functions. However, transient states require further attention.

Fig. 5 presents an average value of p from a numerical experiment of 200 generations of $M = 1000$ agents each. The experiment was performed independently for each value of q

(step 0:01) and α (as in the legend), it is the numerical counterpart of theoretical plots from Fig. 2. The plots are nearly identical – this shows that the analytic approach is correct at this scope. The mathematical difference is that plots in Fig. 2 represent an average of 200 scalar values of p out of (6) while plots in Fig. 5 represent an average of a population of 1000 per generation then averaged for 200 generations.

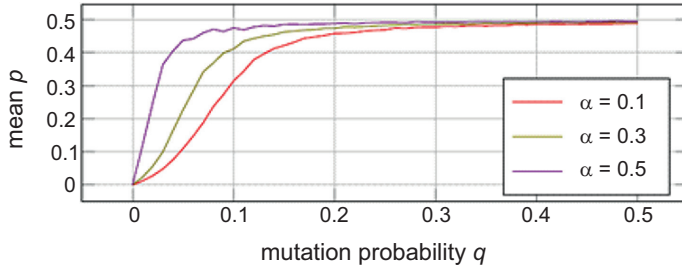


Fig. 5. Mean p as a function of q and α

With closer observation, Fig. 6 and Fig. 7 provide the average values of p for a specific q . The value of p presented in those figures is an average of population of $M = 1000$ in the given generation. These plots are numerical counterparts of plots from Fig. 3 and Fig. 4 respectively. It appears as though plots analytically follow the obtained curves very closely.

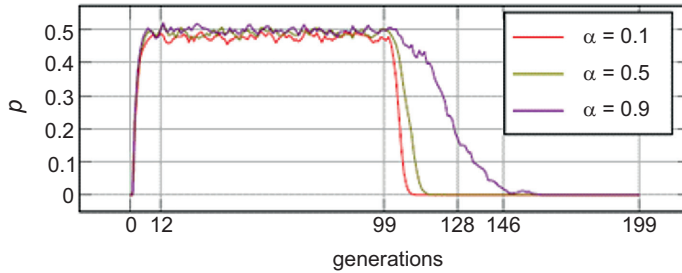


Fig. 6. The results of running the simulation with $q = 0.25$ for generations 0–99 and subsequently, $q = 0.00$

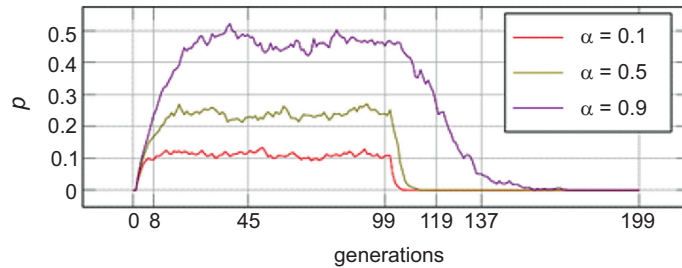


Fig. 7. The results of running the simulation with $q = 0.05$ for generations 0–99 and subsequently, $q = 0.00$

The correlation coefficient between them varies from 93.23% to 99.2% with a median of 98.47%.

6.1. Transient states

In steady states, the numerical results follow the theoretical plots very closely. Additionally, the tendency of change in transient states is correctly predicted, but the exact dynamics of change escapes the theoretical approach from equation (6). Fig. 8 juxtaposes selected corresponding plots from Fig. 3 and Fig. 6 in order to highlight the differences. The interesting part is to the right of generation no. 99, where q was set to 0.0. The plot for $\alpha = 0.1$ shows that for a few iterations, the system behaves as if nothing happened, but the cooperation then falls down so abruptly that it even overtakes theoretical prediction. For $\alpha = 0.9$, the changes also initially ignored but then, p starts to follow the theoretical prediction and falls down very quickly. That last part resembles the theoretical curve for $\alpha = 0.7$, what suggests that the effective α was lower than the nominal α .

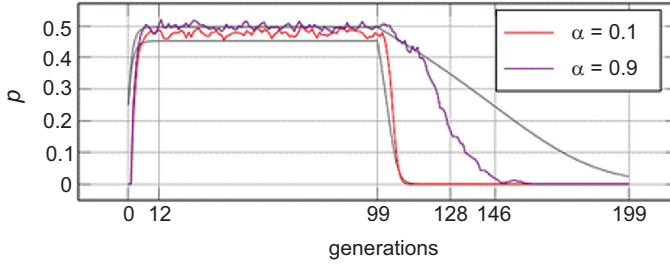


Fig. 8. Comparison of analytical and numerical solutions for $q = 0.25$ in 100 generations and subsequently, $q = 0.0$

We attribute this discrepancy to the fact that the true system simulated here has $M = 1000$ independent agents, each with its own property p , while the theoretical approach is based on a single value of p which mimics what we believe is an average value. The distribution of agents with different values of p is well-balanced, as assumed by theoretical analysis. However, in transient states, such a balance is apparently disturbed. Closer observation is appropriate – firstly, we will divide the population into four classes depending on their

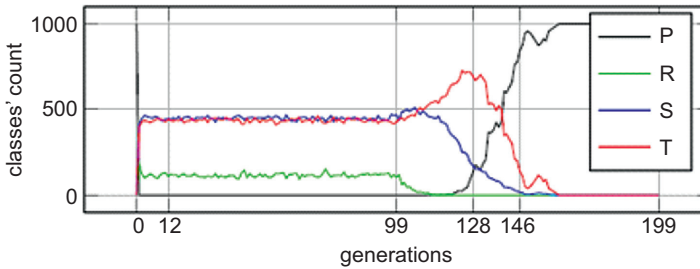


Fig. 9. Classes' count for $q = 0.25$ in 100 generations and subsequently, $q = 0.0$

placement in the process of cooperation. The classes are established in accordance with the nomenclature used in section 3 and named respectively. ‘P’ is the class of agents which do not establish cooperation nor are they included in other agents’ groups; ‘R’ is the class of agents which receive fair returns to their wager; ‘S’ is the class of agents which receive less than fair returns, i.e. they the losers; ‘T’ is the class of agents which receive more than a fair return, i.e. they are the cheats.

Fig. 9 presents the count of classes for $M = 1000$ and initial $q = 0.25$ until the 100th generation. Before discussing the transient part, let us focus on the steady part. As seen, the fraction of fairly treated agents (R) is only about 15% of the total. The remainder is divided roughly in half between those which exploit and those which are exploited. This equilibrium complies with the assumption of a well-mixed population, what proves that the vertex-genetix analysis is correct is this state of the system.

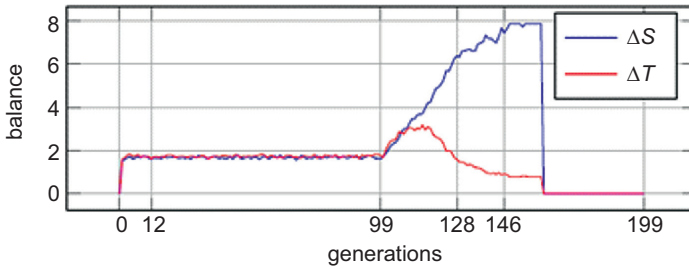


Fig. 10. Balance for $q = 0.25$ in 100 generations and subsequently, $q = 0.0$

In the transient state, where the cooperation is going to disappear, this equilibrium is destroyed, hence the population is no longer well mixed. The number of exploited agents lowers successively, while the number of cheaters grows until it reaches certain point. The intrinsic question is where the extra profit is coming from if the number of givers is diminishing. The answer is provided by Fig. 10. We introduce a variable balance:

$\Delta = |W_C - W_R|$. For cheats, this means how much they gain over and above the fair value; for those who are exploited, this means how much they lose from a fair value. As seen in the figure, the balance for the S class grows up to extreme value where $W_C = 0.0$ – this means that each member of the S class cannot count on any reciprocation. The amount of profit generated by the S class is fully absorbed by the T class. Without any reciprocation, the cooperating agents are dying off. In other words, cooperation becomes a deadly genetic disease.

7. Conclusions

The system presented herein is a new evolutionary game that includes a social dilemma. It emphasises basic evolutionary phenomena: mutation and inheritance, over the strategies exercised in the prisoner’s dilemma and other such games. It also provides another degree of freedom – players are able to establish their own number of participants. The question of fair and unfair cooperations has an answer relative to each player’s wager, that is set as a number of its participants.

Statistical fluctuation introduces mutation into the inheritance. With the initial population of non-cooperating agents, mutation is able to introduce cooperation. As has been shown, mutation is important in keeping the established cooperation, being a counterforce to selection, that prefers cheaters, thereby is a disruptive force for cooperation.

References

- [1] Hardin G., *The tragedy of the commons*, Science 162, 1968, 1243-1248.
- [2] Archetti M., Scheuring I., *Review: Game theory of public goods in one-shot social dilemmas without assortment*, Journal of Theoretical Biology, 299 (0), 2012, 9-20, evolution of Cooperation, <http://www.sciencedirect.com/science/article/pii/S0022519311003171>.
- [3] Hamilton W., *The genetical evolution of social behaviour*, Journal of Theoretical Biology 7 (1), 1964, 1-52, <http://www.sciencedirect.com/science/article/pii/0022519364900384>.
- [4] Archetti M., *The volunteer's dilemma and the optimal size of a social group*, Journal of Theoretical Biology, 261 (3), 2009, 475-480, <http://www.sciencedirect.com/science/article/pii/S0022519309003890>.
- [5] Lee J.-H., Sigmund K., Dieckmann U., Iwasa Y., *Games of corruption: How to suppress illegal logging*, Journal of Theoretical Biology, 367 (0), 2015, 1-13, <http://www.sciencedirect.com/science/article/pii/S0022519314006274>.
- [6] Rauwolf P., Mitchell D., Bryson J.J., *Value homophily benefits cooperation but motivates employing incorrect social information*, Journal of Theoretical Biology, 367 (0), 2015, 246-261, <http://www.sciencedirect.com/science/article/pii/S0022519314006754>.
- [7] Chen X., Gross T., Dieckmann U., *Shared rewarding overcomes defection traps in generalized volunteer's dilemmas*, Journal of Theoretical Biology, 335 (0), 2013, 13-21, <http://www.sciencedirect.com/science/article/pii/S0022519313002865>.
- [8] Grujic J., Cuesta J.A., Sanchez A., *On the coexistence of cooperators, defectors and conditional cooperators in the multiplayer iterated prisoner's dilemma*, Journal of Theoretical Biology, 300 (0), 2012, 299-308, <http://www.sciencedirect.com/science/article/pii/S0022519312000677>.
- [9] Szabo G., Szolnoki A., Czako L., *Coexistence of fraternity and egoism for spatial social dilemmas*, Journal of Theoretical Biology, 317 (0), 2013, 126-132, <http://www.sciencedirect.com/science/article/pii/S0022519312005395>.
- [10] Axelrod R., *The Evolution of Cooperation*, Basic Books, New York 1984.
- [11] Santos M.D., Pinheiro F.L., Santos F.C., Pacheco J.M., *Dynamics of n-person snowdrift games in structured populations*, Journal of Theoretical Biology, 315 (0), 2012, 81-86, <http://www.sciencedirect.com/science/article/pii/S0022519312004699>.
- [12] Qi H., Ma S., Jia N., Wang G., *Experiments on individual strategy updating in iterated snowdrift game under random rematching*, Journal of Theoretical Biology, 368 (0), 2015, 1-12, <http://www.sciencedirect.com/science/article/pii/S002251931400705X>.
- [13] Fu F., Nowak M.A., Hauert C., *Invasion and expansion of cooperators in lattice populations: Prisoner's dilemma vs. snowdrift games*, Journal of Theoretical Biology, 266 (3), 2010, 358-366, <http://www.sciencedirect.com/science/article/pii/S0022519310003395>.
- [14] Baird R.W., Dill L.M., *Ecological and social determinants of group size in transient killer whales*, Behavioral Ecology, 7, 1996, 408-416.
- [15] Moreira J.A., Pinheiro F.L., Nunes A., Pacheco J.M., *Evolutionary dynamics of collective action when individual fitness derives from group decisions taken in the past*, Journal of Theoretical Biology, 298 (0), 2012, 8-15, <http://www.sciencedirect.com/science/article/pii/S0022519311006515>.

- [16] Schoenmakers S., Hilbe C., Blasius B., Traulsen A., *Sanctions as honest signals the evolution of pool punishment by public sanctioning institutions*, Journal of Theoretical Biology, 356 (0), 2014, 36–46, <http://www.sciencedirect.com/science/article/pii/S0022519314002392>.
- [17] Brnnstrm A., Gross T., Blasius B., Dieckmann U., *Consequences of fluctuating group size for the evolution of cooperation*, Journal of mathematical biology, 63 (2), 2011, 263-281.
- [18] Siwik L., Włodarczyk K., Kluczny M., *Staged event-driven architecture as a micro-architecture of distributed and pluginable crawling platform*, Computer Science 14 (4), 2013, 645-665.
- [19] Niedzwiecki M., Cetnarowicz K., Rzecki K., *Using the evaluation nets modeling tool concept as an enhancement of the petri net tool*, [in:] Computer Science and Information Systems (FedCSIS), Federated Conference on, 2013, 1007-1012.
- [20] Kułakowski K., Szybowski J., Tadeusiewicz R., *Tender with success – the pairwise comparisons approach*, Procedia Computer Science 35, 2014, 1122-1131.
- [21] Koczkodaj W., Mackasey W., Smolewski I., Tadeusiewicz R., *How to support the government procurement by a pairwise comparisons model*, Studia Informatica 1–2 (17), 2013, 19-28.
- [22] Tadeusiewicz R., Horzyk A., *Man-machine interactions improvement by means of automatic human personality identification*. chapter in book: Saeed k., snasel v., Computer information systems and industrial management, lecture notes in computer science, Springer-Verlag Berlin Heidelberg 8838, 2014, 278-289.
- [23] Dreżewski R., *Agent-based simulation model of sexual selection mechanism*, [in:] G. Jezic, R.J. Howlett, L.C. Jain (eds.), *Agent and Multi-Agent Systems: Technologies and Applications*, 9th KES International Conference, KES-AMSTA 2015 Sorrento, Italy, June 2015, Proceedings, Vol. 38 of Smart Innovation, Systems and Technologies, 155-166. Springer International Publishing, 2015; The original publication is available at <http://dx.doi.org/10.1007/978-3-319-19728-9>.
- [24] Dreżewski R., *Agent-based modeling and simulation of species formation processes*, [in:] F. Alkhateeb, E. Al Maghayreh, I. Abu Doush, (eds.), *Multi-Agent Systems - Modeling, Interactions, Simulations and Case Studies*, 3-28, [in:] Tech, Rijeka 2011; The original publication is available at <http://dx.doi.org/10.5772/14286>.

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THE DYNAMICS OF THE DISTRIBUTION OF CO-AUTHORS

DYNAMIKA DYSTRYBUCJI WSPÓŁAUTORÓW

Abstract

In the present paper, the case of a database of scientific articles is described. There can be observed the quantitative effects of the increasing strength of cooperation between scholars. It is manifested in time-related features changes of the numbers distribution of a single co-authors' article. The distribution of the number of co-authors of an article recorded in the database evolves with time from a profile with no more than one author to a profile with several authors. A social dilemma model is proposed to explain the dynamics of changes in the distribution of the number of co-authors. The most successful strategy of the three considered alternative strategies of cooperation is selected.

Keywords: cooperation, PubMed, authorship, social dilemma

Streszczenie

W niniejszym artykule, na podstawie bazy artykułów naukowych, opisano ilościowy efekt wzrostu współpracy uczonych, która objawia się zmianami cech rozkładu liczby współautorów artykułu w czasie. Rozkład liczby współautorów artykułu dostępnego w bazie danych zmienia się w czasie z maksimum dla jednego autora do maksimum kilku autorów. Do wyjaśnienia dynamiki zmian rozkładu liczby współautorów zaproponowano model dylematu społecznego. Spośród trzech została wybrana jedna strategia współpracy odnosząca największe sukcesy.

Słowa kluczowe: współpraca, PubMed, autorstwo, dylemat społeczny

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1. Introduction

Cooperation is fundamental to numerous biological and social systems. Multicellular organisms, social insects, herds of animals and human societies are examples of systems that are dependent on cooperative interactions. Explaining the emergence of cooperation has been a challenge since Darwin. Apparently, cooperators help others at a cost to themselves, while defectors receive the benefits of altruism without providing any help in return. The emergence of cooperation has been studied based on tools developed within the framework of evolutionary game theory. Games such as prisoner's dilemma games (PDGs) [1], snowdrift games (SGs) [2] and stag-hunt games (SHs) [3] form the standard set of social dilemmas. All aforementioned evolutionary games are played by a pair of players in which they receive a reward R for mutual cooperation and a punishment P for mutual defection. If one cooperates but the other defects, the defector (D) receives the payoff T , while the cooperator (C) gets the payoff S . In the case of PDG we have $T > R > P > S$. In a SG, the payoff rank is $T > R > S > P$. The stag-hunt game offers more support for cooperation because the rank of payoffs ($R > T > P > S$) favours cooperation over defection. Public goods games (PGGs) [4] engage many players and the number of players is an adjustable parameter. Cooperating players contribute to a common pool, which is then enhanced by a factor η and equally distributed between cooperators and defectors. According to these models, a cooperator pays a price to receive a benefit; conversely, a defector pays no price but can receive extra benefits. It is commonly assumed that cost and benefit are measured in terms of fitness. It transpires that in any mixed population, defectors have a higher average fitness than cooperators and thus, after some time, cooperators vanish from the population. However, it is possible that under some circumstances, cooperators are not extinct in the course of system evolution.

In the present study, the case of a real system of cooperating agents, which adjust their cooperation preferences to the varying competitive pressure is described. As a result, the number of agents engaged in a single cooperation event changes over the course of the evolution of the system. The analysis of the system is based on real-world data – a database compiling bibliographic details of articles published in scientific periodicals over the last 60 years. Publishing an article is a cooperative act and the cooperation preferences of the scholars can be quantitatively characterised by the features of the distribution of the number of co-authors of a single article. From the analysis of the database, it follows that the average number of co-authors increases with time. The modal value of the number of co-authors of a single article increased from 1 to 4 in the period from 1950 to 2010 – this is a clear sign of the increasing strength of cooperation. We propose a novel social dilemma model to explain the observed dynamics of the distribution of the number of co-authors; possible further developments of the model are also discussed.

2. Related works

Mechanisms that facilitate cooperation, include kin selection [5–9], direct and indirect reciprocity [1, 10–13], network reciprocity [14–17], trust [18, 19], group selection [20, 21], cooperation in game theory [22, 23], and evolution of cooperation [24, 25].

An important quantitative aspect of cooperation is the number of players engaged in a single play. The number of players playing the aforementioned social dilemma games is always fixed, while in the case of real systems, the number of players engaged in a single cooperation event is a quantity that should be set by the dynamics of the game. For example, in studies of the behaviour of predators [26–29], the group size is partially determined by food intake, which is maximized for an optimal group size [30]. The problem of an optimal group size has been considered recently in the context of another social dilemma – the volunteer’s dilemma [31], but in that case, the group size is not set dynamically in the course of the system evolution. A mechanism of setting group size, based on ecology-inspired principles, has also been proposed within the context of the public goods game [32, 33]. An emerging field of co-evolutionary games [34, 22] can become a convenient framework for studying the problem of how the output of the games played by the players may affect the interaction network.

The evolution of the research of cooperation could be extended into crawling systems [35], where such cooperation would improve crawling performance. The cooperation processes are able to be modelled using Petri nets [36] to visualise various cooperation mechanisms. Computational intelligence methods [37] appear to be usable for evolution games, they are both based on nature laws and should be combined into one research strategy.

3. Empirical data

The process of development of cooperation was analysed using data collected in the PubMed database (<http://www.ncbi.nlm.nih.gov/pubmed>). PubMed is a free resource, maintained by the National Center for Biotechnology Information (NCBI), at the U.S. National Library of Medicine (NLM). At present, PubMed comprises over 20 million citations for biomedical literature from MEDLINE, life science journals, and online books. PubMed citations and abstracts include the fields of medicine, nursing, dentistry, veterinary medicine, the health care system, and preclinical sciences.

The advanced search functionality of PubMed enables searching for articles that match user-defined patterns based on a set of predefined fields provided by the search engine. The results of a query can be returned in the form of XML files – these can be easily analysed due to their well-defined syntax. In the present study, we queried PubMed for articles published in a single, randomly picked month of the year (April), for years in the range 1950 to 2010, with 5-year intervals. Only records with a complete list of authors and valid authorship data were analysed. Additionally, for every valid record, the scholars who written the article were counted and the distribution of the number of authors of published articles was determined for every analysed year. The distributions of the number of co-authors are shown in Fig. 1. The probability of finding an article published in April 1950, written by more than three authors is equal to around 10%, the modal number of co-authors of a single article is equal to 4 for articles published in April 2010. Articles co-written by less than three authors constitute only about 23% of the total number of articles published in April 2010.

Note that the outcome of standard evolutionary games is the change of the fitness of the players adopting different strategies. By contrast, for the system described above,

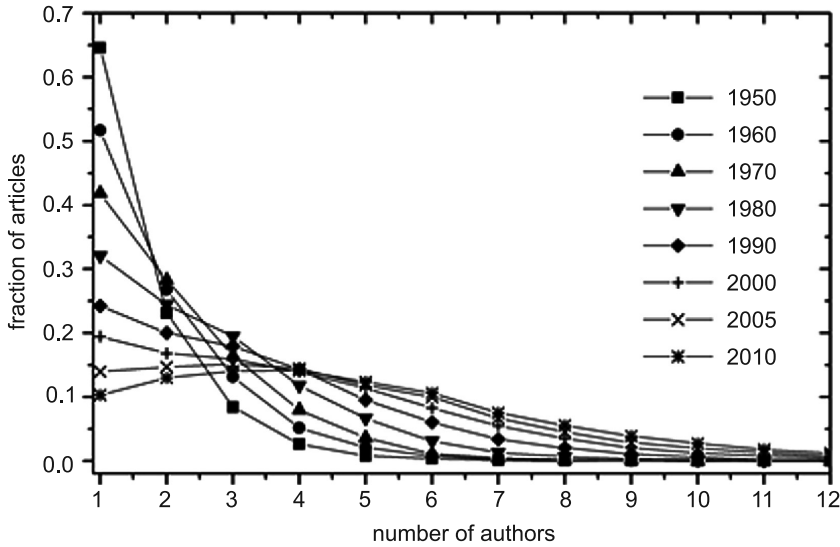


Fig. 1. Empirical distributions of the number of co-authors of a single article

the available quantitative data is the distribution of the number of co-authors and one can only speculate as to how the fitness of the scholars is related to that distribution.

Because it appears that the standard evolutionary models of cooperation (PDGs, SGs, SHs or PGGs) do not have enough features to determine the outcome of cooperation in the form, for example, joint publications, there is a need of the new tool to model the kind of data described above. It is important that possible models account for the competition between scholars for limited resources as the total volume of accessible journals is limited. Additionally, to build their prestige, scholars attempt to prepare publications that are of as high a standard as possible, but an author cannot generally compare the quality of his or her publications to the quality of publications by other authors – this is the task of the reviewers who score the submitted publications.

3.1. Model

To qualitatively explain the empirical data discussed in the previous section, we consider a population consisting of N agents. Each agent A is located on one site of a network G and can cooperate with other agents, provided that they are connected to agent A by an edge in G . Throughout this paper, it is assumed that G is a random graph, although other types of complex network underlying the model are clearly possible.

An agent is an abstract object characterised by the properties and actions which they can perform. The properties of an agent define their state, which is modified in the course of simulation. According to the proposed model, an agent's properties are:

1. credit C
2. activity E

3. quality Q
4. standard deviation of quality sdQ
5. probability of cooperation P
6. list of neighbours L in the underlying network G

An agent can perform two actions. First, they can update their probability of cooperation P . The formula for probability of cooperation update can possibly depend upon an agent's properties (e.g. their actual credit) and defines an agent's strategy of cooperation. Secondly, an agent can compete for resources, submitting proposals for resource allocation. A proposal can be submitted individually or in cooperation with other agents, depending on probability of cooperation P .

The state of the system (defined as the state of all agents) is modified iteratively. At the beginning of the simulations, credit C_A , activity E_A , quality Q_A and standard deviation of quality sdQ_A of an agent A are drawn from normal distributions with the mean and standard deviation being the parameters of the model. Activity E_A , quality Q_A and the standard deviation of quality sdQ_A are kept fixed during the simulations. The probability of cooperation P_A of agent A is initially equal to 0 and is updated in every simulation step in accordance with the cooperation strategy of agent A . The list of neighbours L_A is also fixed – it is assumed that the topology of G does not change in the course of simulation.

At every simulation step, the credit of each agent is decreased by a constant amount dC (which, for example, represents the costs of participating in the game). Then, with probability E_A , every agent generates a proposal for resource allocation. A proposal for resource allocation R is an abstract object characterised by its quality Q_R and the list of agents L_R , which have authorised it. When an agent A generates a proposal R , the quality of proposal Q_R is drawn from a normal distribution with mean and standard deviation equal to Q_A and sdQ_A respectively. The list of agents, which have authorised *see above note* R contains initially only one element – the agent A . Then for every agent B in L_A , the agent A invites the agent B to participate in the proposal R . The invitation is accepted with probability $PA \cdot PB$. After accepting an invitation, the quality Q_R of R is increased by amount q which is drawn from a normal distribution with mean and standard deviation equal to Q_B and sdQ_B respectively. Agent B is then added to L_R .

All agents submit their proposals synchronously and a pool of all submitted proposals is created. The proposals are then scored by their quality and the authors of each proposal receive the payoffs based on the scores of their proposals. Each payoff is shared in equal parts between all authors of a proposal and the credit of each author is increased accordingly. It is assumed in the model that the total sum of payoffs is adjusted so as to keep the total sum of credits constant. In the simulations, it was also assumed that the payoff depends linearly on the rank of a proposal in the pool – the proposal with the lowest quality receives zero payoff, while the proposal with the highest quality receives the highest payoff.

Three strategies of cooperation were studied. Agents adopting the strategy 'Never cooperate' keep their probability of cooperation always equal to zero. Agents adopting the strategy 'Cooperate when in trouble' increase their probability of cooperation whenever their credit decreases, otherwise they decrease their probability of cooperation. Agents adopting the strategy 'Cooperate' update their probability of cooperation whenever their credit decreases but never decrease it below the actual value. Every agent updates their

probability of cooperation in every simulation step. It was assumed that the credit-probability of cooperation dependence is given by a sigmoid-like function approximating between 1 for low credits and 0 for large credits.

4. Social dilemma in the network of co-authors

To see that the model described above belongs to a class of social dilemmas, consider the following simplified version of the model. Let each agent be characterised by the same constant probability of cooperation P and a constant number of cooperators N . Assume for now that P determines the probability that a given agent acquires another agent to authorise his proposal. Let every agent accept the invitation to authorise other agent's proposal with probability equal to 1. It should also be assumed that all agents submit their proposals at every time step and the payoff of the proposal authorised by $k \geq 1$ co-authors is equal to $f(k)$. In the following equation, it is assumed that $f(k) = k^\alpha$, where $\alpha \geq 0$ is an exponent. Then the average payoff S of a single agent can be split into two parts: the payoff S_A from the proposal submitted by the agents and the payoff S_C from the proposals of the agents, which have invited the agent to write their proposals. One has:

$$S_A = \sum_{k=0}^N \binom{N}{k} \cdot P^k \cdot (1-P)^{N-k} \cdot \frac{f(k+1)}{k+1} \quad (1)$$

$$S_C = N \cdot P \cdot \sum_{k=0}^{N-1} \binom{N-1}{k} \cdot P^k \cdot (1-P)^{N-1-k} \cdot \frac{f(k+2)}{k+2} \quad (2)$$

After basic manipulation, it can be shown that:

$$S = S_A + S_C = \sum_{k=0}^N \binom{N}{k} \cdot P^k \cdot (1-P)^{N-k} \cdot f(k+1) \quad (3)$$

It follows immediately from Eq. 1 and 2 that if $f(k) = k^\alpha$, then for any $\alpha \geq 0$ smaller than 1, the payoff S_A of an agent which cooperates ($P > 0$) is smaller than 1, which is the payoff S_A of a defector ($P = 0$). Consequently, a defector facing cooperators would have higher fitness than his cooperating co-players; thus, a rational player will choose the strategy to defect. On the other hand, it follows from Eq. 3 that the average payoff (equal to 1) in a network of defectors is lower than the average payoff in a network of cooperators. Thus, in the described system, the agents are facing a social dilemma. Note that the cooperation strategy is always advantageous if $\alpha > 0$.

To solve the social dilemma described above, an additional constraints must be introduced to promote cooperation. In the present study, the reciprocity mechanism is adopted. In the case described above, cooperation cannot emerge because the probability P of acquiring a co-author is independent on the probability (equal to 1) of accepting other agents' invitations to authorise their proposals. If each agent is instead characterised by

a single probability of cooperation which determines both the probability of proposing authorship to other agents and the probability of accepting other agents invitations, then cooperation emerges, as is described in the next section.

4.1. Results

The results presented in the current section were obtained for the following values of model parameters. A random network of 1000 agents was created with the average number of neighbours being equal to 20 (the total number of network links was 10 times larger than the number of agents). The initial credit of the agents was drawn from normal distribution $N(30, 2)$ with the mean and standard deviation equal to 30 and 2 respectively. At every simulation step, the credit was decreased by 0.1. The quality of agents was drawn from normal distribution $N(2, 0.5)$. The standard deviation of quality was set to 0.2 and the activity was set to 0.5 for all agents. The credit-probability of cooperation dependence was given by the function:

$$P(C) = \frac{1}{1 + e^{C/10}} \quad (4)$$

where P stands for the probability of cooperation and C stands for an agent's credit.

The credit of agents adopting the strategy 'Never cooperate' is plotted against their quality in Fig. 2 for three points in time. Note that the destiny of an agent is completely determined by their quality; thus, in a sense, an agent has no way to escape their destiny. Inequality of credit share or social inequality increases over time. The distribution of credit values is more or less uniform after an initial transient period (Fig. 3).

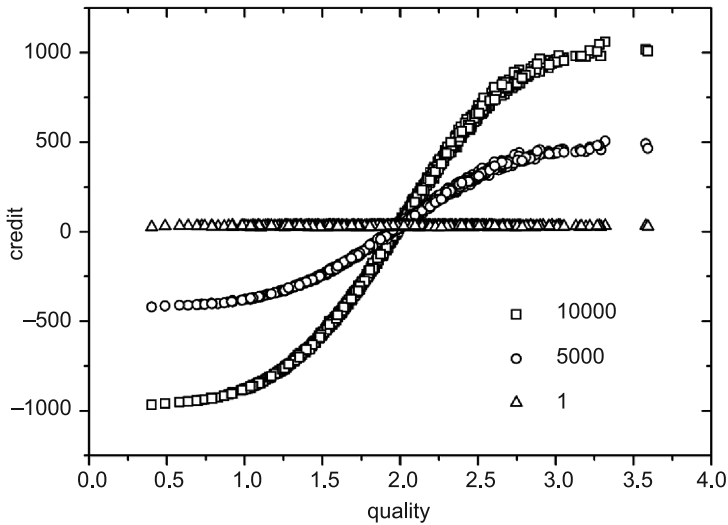


Fig. 2. Credit plotted against the quality of agents in a network of agents adopting the strategy 'Never cooperate'. The results for time (measured in simulation steps) equal to 1 unit (triangles), 5,000 units (circles) and 10,000 units (squares)

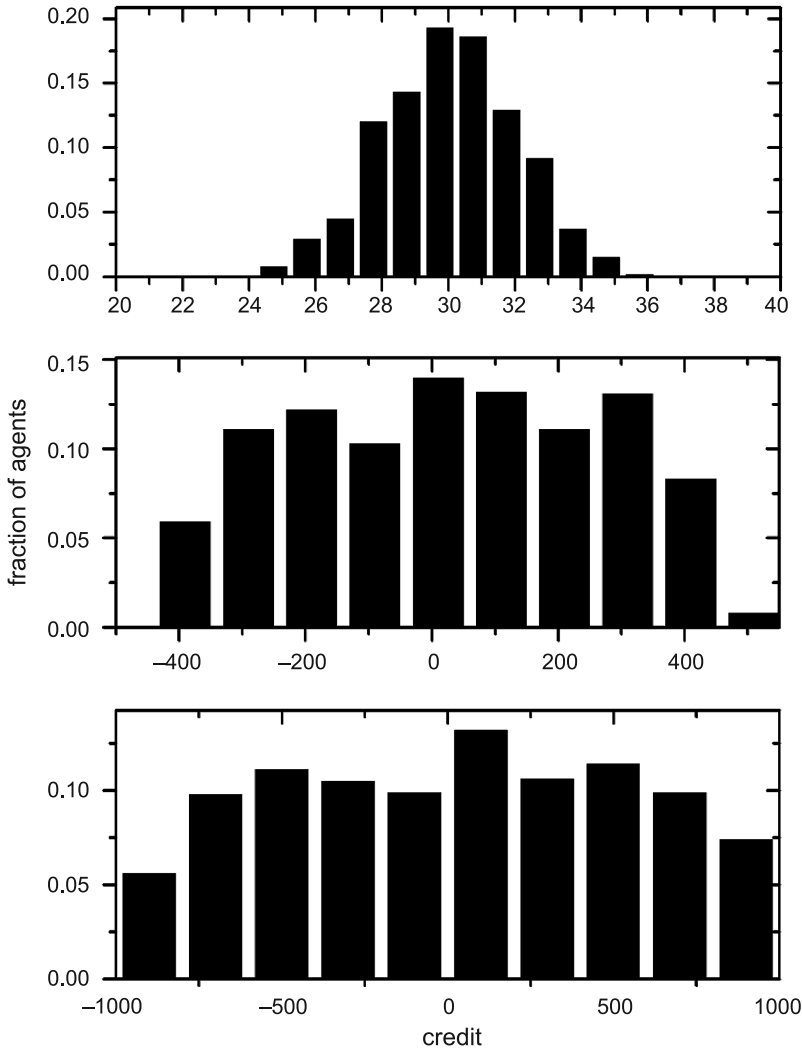


Fig. 3. The distribution of credits in a network of agents adopting the strategy 'Never cooperate'. The results for time (measured in simulation steps) equal to 1 unit (top panel), 5,000 units (middle panel) and 10,000 units (bottom panel)

The credit of agents adopting the strategy 'Cooperate when in troubles' is plotted against their quality in Fig. 4 for four points in time. Also in this case a strong dependence between an agent's quality and their credit is observed in some range of quality values, but that range shrinks with time. Comparison of Fig. 4 and Fig. 5, in which probability of cooperation is plotted vs. an agent's quality indicates that that strong dependence is observed for agents which actually do not cooperate, keeping their probability of cooperation equal to zero. The group of non-cooperating agents disappears with time and a more or less egalitarian

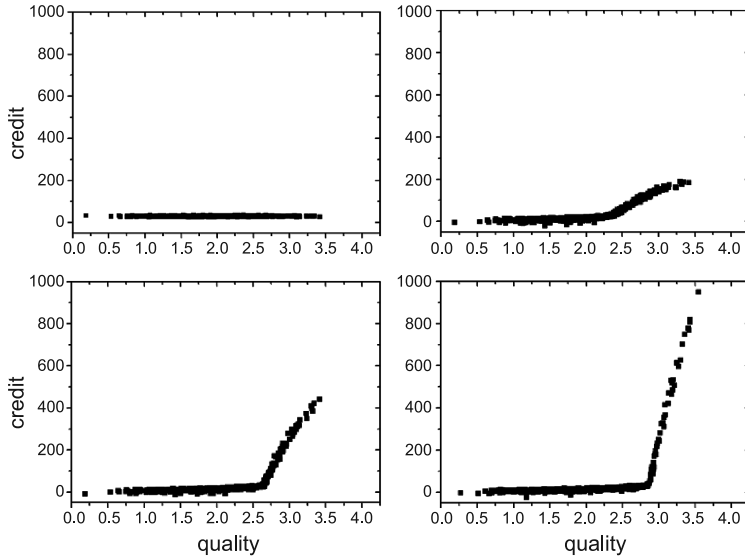


Fig. 4. Credit plotted against the quality of agents in a network of agents adopting the strategy ‘Cooperate when in trouble’. The results for time (measured in simulation steps) equal to 1 unit (top-left panel), 5,000 units (top-right panel), 10,000 units (bottom-left panel) and 30,000 units (bottom-right panel)

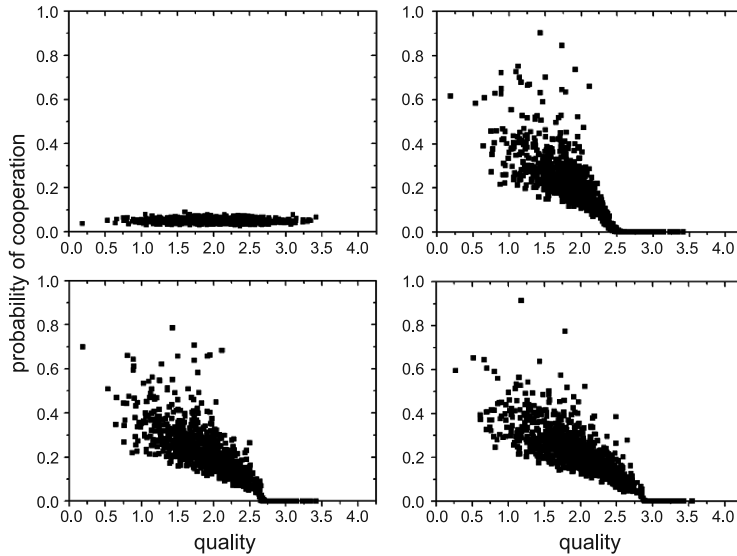


Fig. 5. Probability of cooperation plotted against the quality of agents in a network of agents adopting the strategy ‘Cooperate when in troubles’. The results for time (measured in simulation steps) equal to 1 (top-left panel), 5,000 units (top-right panel), 10,000 units (bottom-left panel) and 30,000 units (bottom-right panel)

society arises (Fig. 6). Although cooperation becomes yet more important in a network of agents adopting the strategy ‘Cooperate when in troubles’, the distribution of the number of co-authors of a single proposal converges to a profile which has a maximum for one author (Fig. 7). The average number of co-authors of a single proposal converges to 1.78 ± 0.04 (Fig. 8).

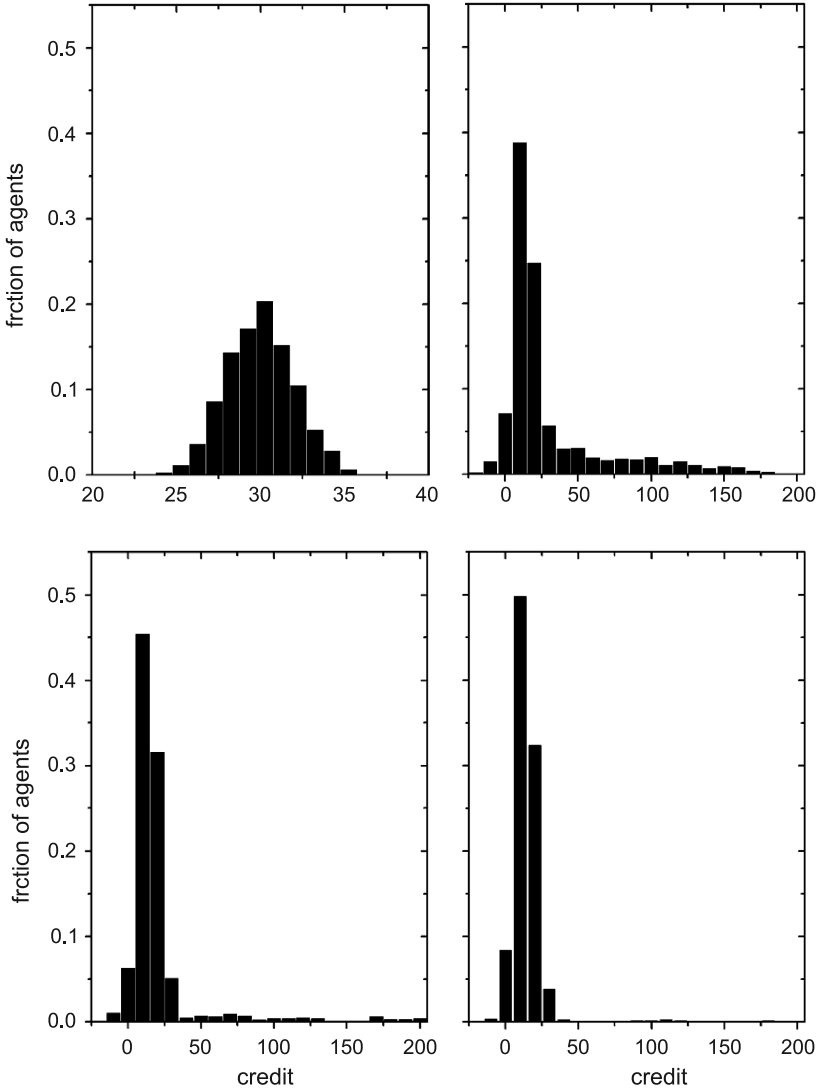


Fig. 6. The distribution of credits in a network of agents adopting the strategy ‘Cooperate when in trouble’. The results for time (measured in simulation steps) equal to 1 unit (top-left panel), 5,000 units (top-right panel), 10,000 units (bottom-left panel) and 30,000 units (bottom-right panel)

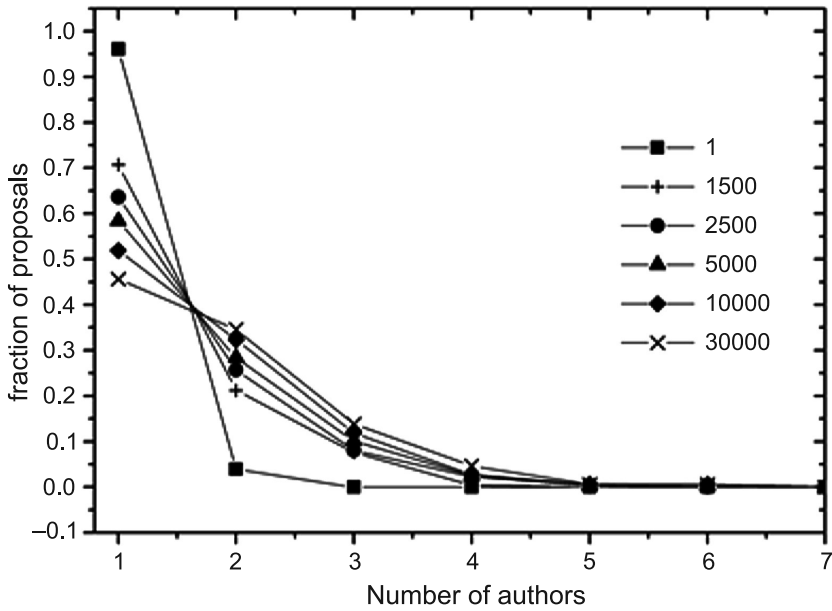


Fig. 7. Model distributions of the number of co-authors of a single article for the case of agents adopting the strategy ‘Cooperate when in troubles’

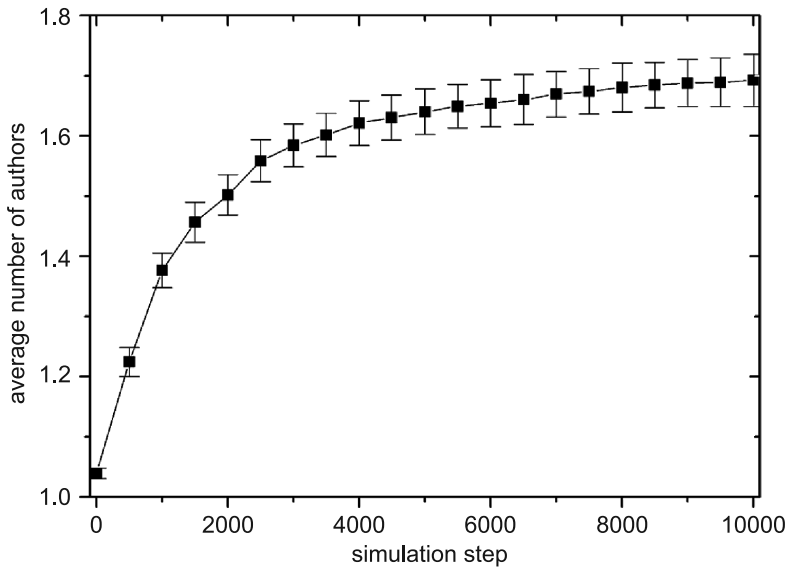


Fig. 8. Average number of the co-authors of a single article plotted against the simulation step ‘time’ for the case of agents adopting the strategy ‘Cooperate when in trouble’. The error bars represent the mean square error of the average number of authors

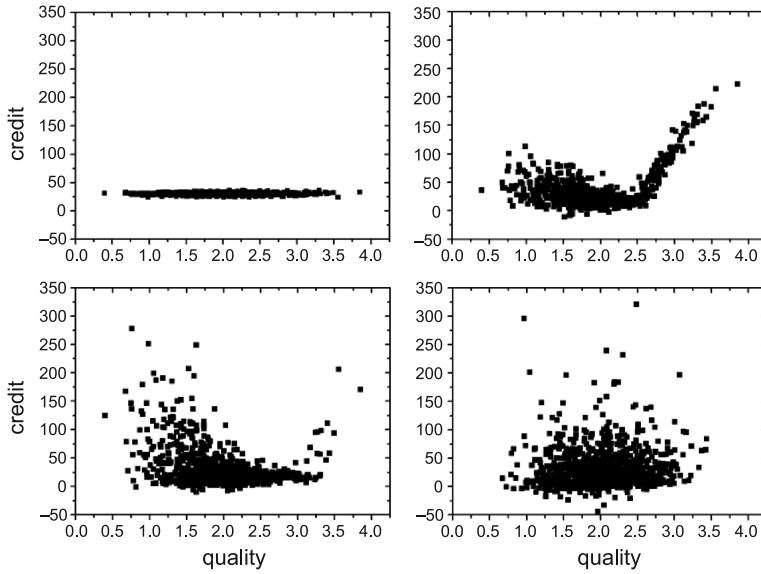


Fig. 9. Credit plotted against the quality of agents in a network of agents adopting the strategy 'Cooperate'. The results for time (measured in simulation steps) equal to 1 unit (top-left panel), 5,000 units (top-right panel), 10,000 units (bottom-left panel) and 80,000 units (bottom-right panel)

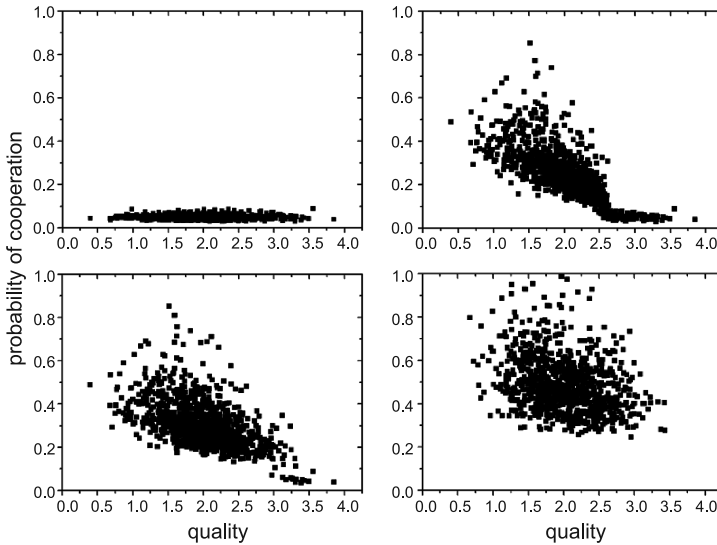


Fig. 10. Probability of cooperation plotted against the quality of agents in a network of agents adopting the strategy 'Cooperate'. The results for time (measured in simulation steps) equal to 1 unit (top-left panel), 5,000 units (top-right panel), 10,000 units (bottom-left panel) and 80,000 units (bottom-right panel)

The credit of agents adopting the strategy ‘Cooperate’ is plotted against their quality in Fig. 9 for four points in time. Relatively early, two branches develop on the quality-credit plot. The branch observed for high values of quality is the trace of the existence of the high-quality agents which have not yet decided to actively cooperate, thus keeping their probability of cooperation low (Fig. 10). On the other hand, can be observed a branch

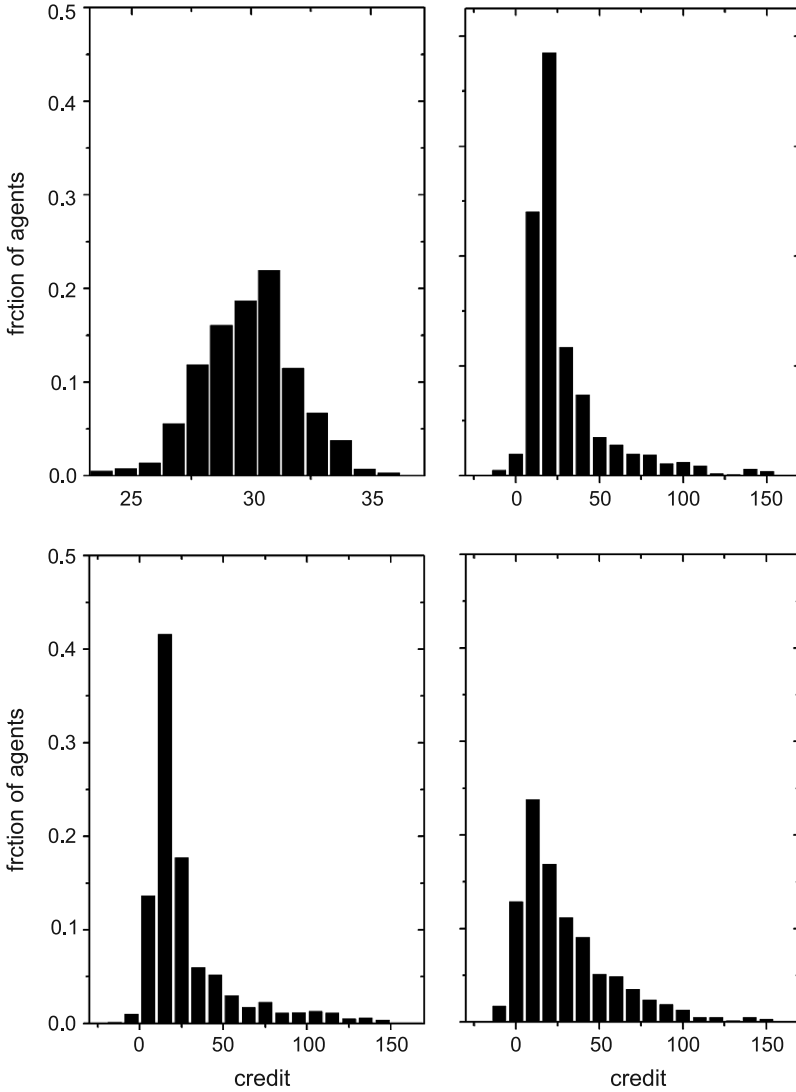


Fig. 11. The distribution of credits in a network of agents adopting strategy ‘Cooperate’. The results for time (measured in simulation steps) equal to 1 unit (top-left panel), 5,000 units (top-right panel), 10,000 units (bottom-left panel) and 80,000 units (bottom-right panel)

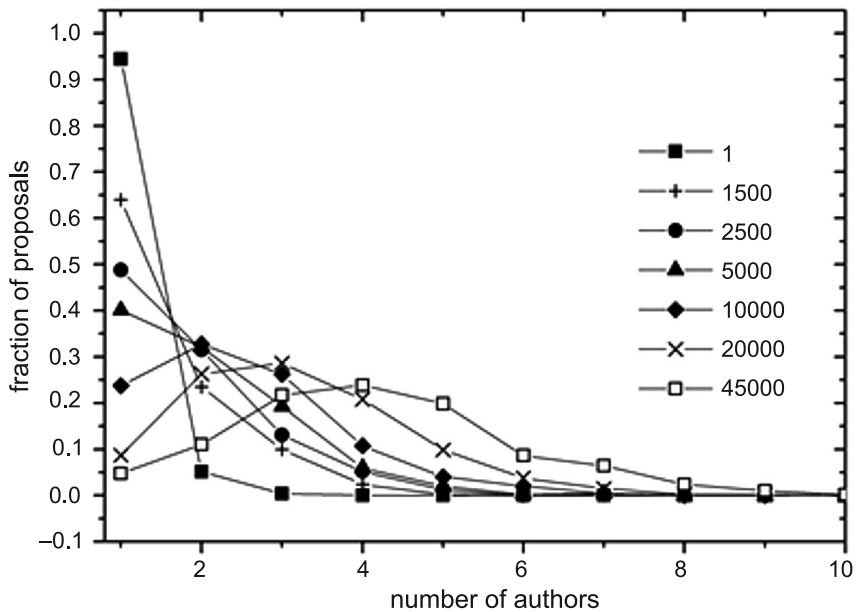


Fig. 12. Model distributions of the number of co-authors of a single article for a case of agents adopting the strategy 'Cooperate'

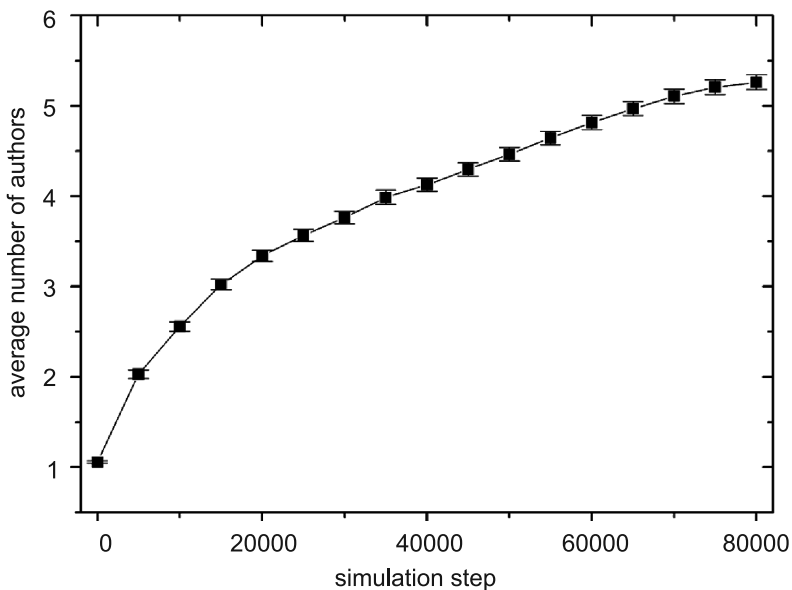


Fig. 13. Average number of the co-authors of a single article plotted against simulation step 'time' for a case of agents adopting the strategy 'Cooperate'. The error bars represent the mean square error of the average number of authors

in the range of low qualities is the trace of the action of low quality agents, that cooperate and join their attempts to gain a higher share in the resource allocation. After a long time the credit of agents is decoupled from their quality and the cooperation is the main mechanism, which determines an agent's credit. The distribution of the agents' credits is broader than in the case of agents adopting the strategy 'Cooperate when in trouble' and has an exponential tail (Fig. 11). Because of high competitive pressure (agents never decrease their probability of cooperation) in a network of agents adopting the strategy 'Cooperate', the modal value of the distribution of the number of co-authors of a single proposal moves to the right in time (Fig. 12). Obviously, there exists a natural limit for the evolution of this distribution, namely, the distribution of the neighbours in the underlying network of contacts G . The average number of co-authors of a single proposal increases with time (Fig. 13).

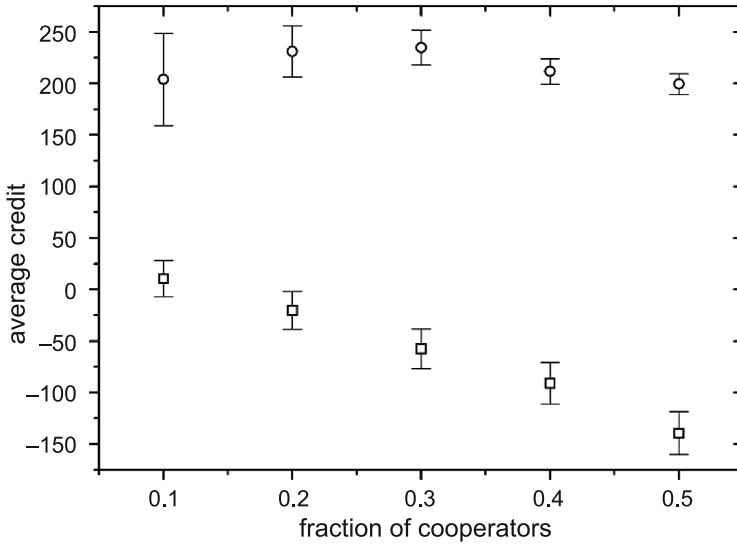


Fig. 14. Average credit of agents adopting the strategy 'Never cooperate' (squares) and 'Cooperate' (circles) in a network consisting of agents of both types plotted against the fraction of agents adopting the strategy 'Cooperate'. The error bars represent the mean square error of the average credit values

Finally, we studied competition between agents adopting different strategies within the same network of contacts. Even for the fairly small fraction of agents adopting the strategy 'Cooperate', these agents gain a much higher share of the total credit compared to agents adopting the strategy 'Never cooperate' (Fig. 14). Similar results (not shown) were obtained for networks consisting of agents adopting the strategies 'Cooperate' and 'Cooperate when in trouble'.

5. Discussion

In the present paper, we have described the case of a database of scientific articles in which the emergence of cooperation between the scholars and the increasing strength of cooperation between them is manifested by an increasing average number of co-authors of a single article, recorded in the database. We have also proposed a social dilemma model to explain the dynamics of the changes in the distribution of the number of co-authors. Among three considered play strategies, the most successful strategy was selected.

Within the framework of the proposed model, the agents compete for common, limited resources, which are allocated to them based on their quality (or fitness). Based on the acquired fitness, the agents adjust their behaviour. Commonly, a generation exchange is implemented within the framework of evolutionary games – this feature is not embedded in the presented model because the time scale of the dynamics of the average number of co-authors observed in the PubMed database is of the same order as the typical duration of an academic career.

The proposed model demonstrates the mechanisms which can induce changes in the distribution of the number of co-authors. It is worth noticing that the dynamics exhibited by the system of agents competing for resources and developing cooperation in the course of simulation is an example of Red Queen dynamics [38]. The agents with low quality must cooperate in order to be allocated the resources they request, otherwise, the resources would be allocated to high quality agents. After a time, groups of cooperating low-quality agents become more effective in gaining resources; thus, high-quality agents are forced to cooperate under increasing competition pressure. However, this has an effect on the low-quality agents, which must strengthen their cooperation.

Many further developments are possible. First, in the present model there is only one probability of cooperation, which determines both the probability of inviting others to cooperate and the probability of accepting cooperation. In reality, the probability of invitation can be different from the probability of accepting invitations. Next, the probability of cooperation can be associated not with an agent but with an arc of the graph of contacts – there is no need to assume an undirected graph in this case because an agent can prefer cooperation with one of his neighbours over cooperation with another individual. In that case, the probability of cooperation can depend on the history of joint publication (successful publication increases probability of cooperation, while a fault decreases it) and on the quality of both sides the link. Next, the network structure was static in the present model, but in reality, it can change at the time scale during the simulations – the number of agents, the number of links and the topology of the graph can evolve over time. Additionally, the resources and the strategy for allocating the resources are fixed and this assumption can be easily relaxed. Finally, the proposed model can be reformulated within the evolutionary framework with credit interpreted as fitness. Another dynamic variable is the function which specifies how the payoff depends on the number of players. In the present paper, a simple form of a function was assumed but there are other choices possible, leading to the formation of stable groups. This problem is currently under study.

The dilemma one is facing when investigating models like PDG is how cooperation arises in the population of selfish individuals. In the model described here, the individuals cooperate because they are selfish – to gain the best resources for themselves, they must create alliances.

References

- [1] Axelrod R., Hamilton W., *The evolution of cooperation*, Science 211, 1981, 1390-1396.
- [2] Sugden R., *The Economics of Rights, Cooperation, and Welfare*, Blackwell, Oxford, England 1986.
- [3] Skyrms B., *Stag-Hunt Game and the Evolution of Social Structure*, Cambridge University Press, Cambridge 2004.
- [4] Hardin G., *The tragedy of the commons*, Science 162, 1968, 1243-1248.
- [5] Hamilton W., *The genetical evolution of social behaviour*, Journal of Theoretical Biology 7 (1), 1964, 1-52, <http://www.sciencedirect.com/science/article/pii/0022519364900384>.
- [6] Taylor P.D., *Altruism in viscous populations an inclusive fitness model*, Evol. Ecol. 6, 1992, 352-356.
- [7] Queller D.C., *Quantitative genetics, inclusive fitness, and group selection*, Am. Nat. 139, 1992, 540-558.
- [8] Frank S.A., *Foundations of Social Evolution*, Princeton Univ. Press, Princeton 1998.
- [9] West S.A., Pen I., Griffin A.S., *Cooperation and competition between relatives*, Science 296, 2002, 72-75.
- [10] Sethi R., Somanathan E., *Understanding reciprocity*, Journal of Economic Behavior & Organization 50 (1), 2003, 1-27, <http://www.sciencedirect.com/science/article/pii/S016726810200032X>.
- [11] Carpenter J.P., *Trust and reciprocity: Interdisciplinary lessons from experimental research: Elinor Ostrom*, James Walker (eds.); Russell Sage Foundation, New York, 2003, xiii and 409 pages, index, Journal of Economic Behavior & Organization 55 (1), 2004, 93-99, <http://www.sciencedirect.com/science/article/pii/S0167268104000319>.
- [12] Trivers R., *The evolution of reciprocal altruism*, Q. Rev. Biol., 46, 1971, 35-57.
- [13] Axelrod R., *The Evolution of Cooperation*, Basic Books, New York 1984.
- [14] Charness G., Haruvy E., Sonsino D., *Social distance and reciprocity: An internet experiment*, Journal of Economic Behavior & Organization, 63 (1), 2007, 88-103. doi: <http://dx.doi.org/10.1016/j.jebo.2005.04.021>. URL <http://www.sciencedirect.com/science/article/pii/S0167268106000564>.
- [15] Borcherting T.E., Filson D., *Group consumption, free-riding, and informal reciprocity agreements*, Journal of Economic Behavior & Organization, 47 (3), 2002, 237-257, <http://www.sciencedirect.com/science/article/pii/S0167268101002062>.
- [16] Neugebauer T., Poulsen A., Schram A., *Fairness and reciprocity in the hawkdove game*, Journal of Economic Behavior & Organization, 66 (2), 2008, 243-250, <http://www.sciencedirect.com/science/article/pii/S0167268106002290>.
- [17] Abbink K., Irlenbusch B., Renner E., *The moonlighting game: An experimental study on reciprocity and retribution*, Journal of Economic Behavior & Organization, 42 (2), 2000, 265-277, <http://www.sciencedirect.com/science/article/pii/S0167268100000895>.
- [18] Cagno D.D., Sciubba E., *Trust, trustworthiness and social networks: Playing a trust game when networks are formed in the lab*, Journal of Economic Behavior & Organization, 75 (2), 2010, 156-167, <http://www.sciencedirect.com/science/article/pii/S0167268110000739>.

- [19] Annen K., *Social capital, inclusive networks, and economic performance*, Journal of Economic Behavior & Organization, 50 (4), 2003, 449-463, <http://www.sciencedirect.com/science/article/pii/S0167268102000355>.
- [20] van den Bergh J.C., Gowdy J.M., *A group selection perspective on economic behavior, institutions and organizations*, Journal of Economic Behavior & Organization, 72 (1), 2009, 1-20, <http://www.sciencedirect.com/science/article/pii/S0167268109001255>.
- [21] Henrich J., *Cultural group selection, coevolutionary processes and large-scale cooperation*, Journal of Economic Behavior & Organization, 53 (1), 2004, 3-35, evolution and Altruism, <http://www.sciencedirect.com/science/article/pii/S0167268103000945>.
- [22] Outkin A.V., *Cooperation and local interactions in the prisoners' dilemma game*, Journal of Economic Behavior & Organization, 52 (4), 2003, 481-503, <http://www.sciencedirect.com/science/article/pii/S0167268102001555>.
- [23] Dreber A., Fudenberg D., Rand D.G., *Who cooperates in repeated games: The role of altruism, inequity aversion, and demographics*, Journal of Economic Behavior & Organization, 98 (0), 2014, 41-55, <http://www.sciencedirect.com/science/article/pii/S0167268113003053>.
- [24] Mullins D.A., Whitehouse H., Atkinson Q.D., *The role of writing and recordkeeping in the cultural evolution of human cooperation*, Journal of Economic Behavior & Organization, 90, Supplement (0), 2013, S141-S151, evolution as a General Theoretical Framework for Economics and Public Policy, <http://www.sciencedirect.com/science/article/pii/S0167268112002764>.
- [25] Janssen M.A., Rollins N.D., *Evolution of cooperation in asymmetric commons dilemmas*, Journal of Economic Behavior & Organization, 81 (1), 2012, 220-229, <http://www.sciencedirect.com/science/article/pii/S0167268111002599>.
- [26] Packer C., Scheel D., Pusey A.E., *Why lions form groups: Food is not enough*, The American Naturalist 136, 1990, 1-19.
- [27] Baird R.W., Dill L.M., *Ecological and social determinants of group size in transient killer whales*, Behavioral Ecology 7, 1996, 408-416.
- [28] Yip E., Powers K.S., Aviles L., *Cooperative capture of large prey solves scaling challenge faced by spider societies*, Proc. Natl. Acad. Sci. USA 105, 2008, 11818-11822.
- [29] Creel S., Creel N.M., *Communal hunting and pack size in african wild dogs*, Lycaon pictus. Anim. Behav. 50, 1995, 1325-1339.
- [30] Jones G., *Are smarter groups more cooperative? evidence from prisoner's dilemma experiments*, 1959-2003, Journal of Economic Behavior & Organization, 68 (3-4), 2008, 489-497, <http://www.sciencedirect.com/science/article/pii/S0167268108001315>.
- [31] Archetti M., *The volunteer's dilemma and the optimal size of a social group*, Journal of Theoretical Biology, 261 (3), 2009, 475-480, <http://www.sciencedirect.com/science/article/pii/S0022519309003890>.
- [32] Hauert C., Wakano J., Doebeli M., *Ecological public goods games: Cooperation and bifurcation*, Theor Pop Biol., 73, 2008, 257-263.
- [33] Charness G., Yang C.-L., *Starting small toward voluntary formation of efficient large groups in public goods provision*, Journal of Economic Behavior & Organization, 102 (0), 2014, 119-132, <http://www.sciencedirect.com/science/article/pii/S0167268114000754>.
- [34] Perc M., Szolnoki A., *Coevolutionary games-a mini review*, BioSystems, 99, 2010, 109-125.
- [35] Siwik L., Włodarczyk K., Kluczny M., *Staged event-driven architecture as a micro-architecture of distributed and pluginable crawling platform*, Computer Science, 14 (4), 2013, 645-665.
- [36] Niedzwiecki M., Cetnarowicz K., Rzecki K., *Using the evaluation nets modeling tool concept as an enhancement of the petri net tool*, [in:] Computer Science and Information Systems (FedCSIS), 2013 Federated Conference on, 2013, 1007-1012.

- [37] Tadeusiewicz R., Horzyk A., *Man-machine interactions improvement by means of automatic human personality identification*, [in:] Saeed K., Snašel V., *Computer information systems and industrial management, lecture notes in computer science*, Springer-Verlag Berlin Heidelberg 8838, 2014, 278-289.
- [38] Van Valen L., *A new evolutionary law*, *Evol. Theory*, 1, 1973, 1-30.

PIOTR ZABAWA*

CONTEXT-DRIVEN META-MODELING FRAMEWORK (CDMM-F) – CONTEXT ROLE

FRAMEWORK METAMODELOWANIA STEROWANEGO KONTEKSTEM (CDMM-F) - ROLA KONTEKSTU

Abstract

This paper introduces an implementation of the Context-Driven Meta-Modeling Paradigm (CDMM-P). This implementation is the proof of concept for the CDMM-P as it shows that the idea of the CDMM-P is feasible. The software system presented here takes the form of the Context-Driven Meta-Modeling Framework (CDMM-F). This framework plays the key role in the Context-Driven Meta-Modeling Technology (CDMM-T) dedicated to generating software system artifacts in a model-driven manner. In contrast to all contemporary approaches to meta-modeling, the meta-model is dynamically loaded from the application context file. In result, the framework has a self-organizing structure and the modeling language does not have a fixed hardcoded predefined structure. This structure, as well as the set of meta-model elements, plays the role of the parameter for the general modeling tool, a part of which is presented in this paper.

Keywords: modeling language, meta-model, application context, Spring, aspect oriented programming, AspectJ

Streszczenie

W artykule wprowadzono implementację Context-Driven Meta-Modeling Paradigm (CDMM-P). Implementacja ta stanowi potwierdzenie poprawności koncepcji CDMM-P, pokazując, że pomysł CDMM-P jest realizowalny. Przedstawiony tu system softwarowy to Context-Driven Meta-Modeling Framework (CDMM-F). Pełni on kluczową rolę w Context-Driven Meta-Modeling Technology (CDMM-T) przeznaczonej do generowania artefaktów systemów softwarowych w podejściu sterowanym modelem. W odróżnieniu od wszystkich współczesnych sposobów metamodelowania, metamodel jest ładowany dynamicznie z pliku kontekstu aplikacji. W efekcie framework ma samoorganizującą się strukturę, a język modelowania nie ma ustalonej i zapisanej w kodzie źródłowym predefiniowanej struktury. Struktura ta i zbiór elementów meta-modelu pełnią rolę parametru dla ogólnego narzędzia modelowania, którego część przedstawiono w niniejszym artykule.

Słowa kluczowe: język modelowania, metamodel, kontekst aplikacji, Spring, programowanie aspektowe, AspectJ

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1. Introduction

There are several approaches to meta-modeling (defining modeling languages) known from scientific and industrial literature. The first approach assumes a fixed structure to the modeling language. In such a case, a particular modeling tool is dedicated to the application of one or several predefined modeling languages. This group of tools is represented by commercial solutions offered by IBM/Rational, Visual Paradigm, No Magic and many others. Some of these have free versions; however, these have limitations. There are also open-source solutions like, for example, Eclipse MDT. The second approach is focused just on meta-modeling [11]. This is represented by such commercial products as MagicDraw [10] or such open-source products as Eclipse EMF [1, 2, 5]. However, these tools are based on predefined and, as a consequence, limited elements that can be applied for the construction of the modeling language. Moreover, the result of the modeling process is a tool (appropriate plug-in) that is able to support the defined modeling language only.

It is worth noticing that there are also several modeling standards supported by the Object Management Group (OMG) [8]. These standards result from the long-term analyses of reasons for project failures which was initiated in the USA in the 1970s. This process led to the elaboration of dozens of software development methodologies from the 1980s and early 1990s. On the basis of these methodologies the Unified Modeling Language (UML) standard [8, 9] was born in Rational, and then was acquired by OMG for standardization in 1997. After that, such standards as the Meta-Object Facility (MOF) [7] and Model-Driven Architecture (MDA) [4] built on top of them were defined at the beginning of the twenty-first century. All of these standards and appropriate tools evolve very dynamically, but they introduced important limits to the possible approaches to constructing modeling languages from the very beginning. The most important factor is that the modeling language customization possibilities are very limited. The user of the modeling language can customize the UML language through UML Profiles but he/she is not able to introduce changes to the language structure. Another important limit introduced to the MOF+UML concept is the nature of the standardization process itself. This process is very long lasting, so when new technology is born, its potential user must wait many years for the acceptance of the new version of the UML standard enriched in the notions introduced by the new technology. This problem is known from such old technologies as Aspect Oriented Programming (AOP) [3] which was born in 2002 and was not yet accommodated by UML. It is also known, from such new technologies as Scala programming language [6], which introduced new relationship – traits. This feature of standards and the unacceptably long standardization process are in opposite to the main assumption of MDA – the ability of fast accommodating new technologies for the purpose of generating software project artifacts implemented in these new technologies.

In this paper, the new software framework is introduced. It is based on the Context-Driven Meta-Modeling Paradigm [12] – the new approach to meta-modeling. This approach can break the limits introduced into MOF&UML concepts. It results in the implementation of the general modeling tool that is able to support all current modeler requirements. In comparison to existing modeling tools, the new software framework provides extra opportunity to model in any user-defined modeling language. As a consequence, the modeler may reuse his/her previous modeling languages (both meta-model classes and relationships) and introduce any

structural changes to them according to the current demands. Another new option for the modeler is to customize the modeling language in the same manner as the one required when vertical OMG standards are applied for a particular application domain. The vertical standards being the superset of system of notions required by a modeler should be customized by the modeler to the needs of a particular software system. Moving this approach to the meta-modeling field, the modeler may customize an existing modeling language standard (such as UML) to any substructure of it. There are no tools on the market that offer such a possibility except from the tool which is introduced here.

2. Problem solution concept

In order to construct the implementation of dynamic loading of any required meta-model to the modeling tool, the right system of notions must be introduced. This problem is addressed in this section.. The concept of the solution to the problem is also presented here.

In contrast to the approaches presented above, the new approach introduced here results in the implementation of one general modeling tool for any modeling language. The user of the tool can define his/her own modeling language in the form of a graph composed of any kind of nodes and any kind of edges. This potentially unlimited language can be loaded to the tool from the meta-model file.

The solution concept is based on the observation that all contemporary known approaches to meta-modeling have a common disadvantage – namely, that the relationships between meta-model classes are represented in the form of hardcoded references. However, it is possible to represent them in the form of dynamically injected responsibilities. Thus, each meta-model can be decomposed into classes (pure entity classes) and relationship classes (entity relationship classes). They can remain unrelated on the level of the compiled source code. The meta-model can be created at run time from dynamically loaded meta-model structure representation. This meta-model structure takes the role of the modeling application context. As a result, the modeling framework structure can reflect the application context. Moreover, the relationships injected into meta-model classes can take the role of one of many possible responsibilities the meta-model classes can dynamically take – this is why the ‘responsibility’ term is frequently used for relationships in the rest of the paper.

3. Tool implementation

In this section, the implementation of the context-driven meta-modeling tool is presented. The tool is dedicated to construction and usage of a model which is compatible to the particular meta-model.

The implemented tool has the form of the framework. It is composed of fixed elements belonging to the framework, such as configuration, API classes and the distinguished meta-model root element. However, the framework is able to construct itself from the meta-model specification. The most natural meta-model specification is the XML file containing this model. This file takes the role of the application context.

In order to implement the framework, some research was performed. The most important programming language for the framework is Java, as all or almost all contemporary modeling tools are implemented in Java or at least expose their APIs for this programming language. The most attractive technology for this programming language is the Spring framework which supports the required dynamic injections and application context. But the Spring alone was recognized as insufficient. The lacking feature – the ability to introduce dynamic structural changes into classes was offered by AspectJ. Samples illustrating the method of combination of these technologies are presented in Section 4.

All these technologies made it possible to implement the framework briefly characterized in Section 2.

4. Testing

Model-driven tools are difficult to test. The reason for this is that they are able to load a potentially infinite number of different data structures, so they cannot be tested against each of these data structure. However, in the context of the problem stated in the paper, the purpose of testing is not to perform commercial verification of the tool, but to focus on verification of the correctness of the tool implementation against the ability of loading the meta-model dynamically. Therefore, this section is dedicated to verifying whether the tool makes possible to:

- load several different meta-models,
- create models compatible to each meta-model through the tool's API,
- navigate through the model structure and acquire model elements.

In order to check each of the functional features presented above, the following steps were performed:

- testing meta-models were created in the form of UML diagrams,
- testing simple XML graphs of these meta-models were created,
- testing Spring application contexts XML files for the framework for the above graphs were created,
- test scenarios for:
 - creating models,
 - navigating through each model structure,
 - getting model elements and checking if they are the same as created before.

In order to illustrate the testing approach, which is done in Subsection 4.1, an example meta-model was chosen. The test scenario for this meta-model is also presented below to display the nature of the client's code. Moreover, the different meta-model structures that were the subject of testing activities are shown below in Subsection 4.1.

4.1. Different meta-model examples

Several different meta-model UML diagrams that are the subject of testing are presented below. They are very simple as they were intended to test meta-model relationship class and meta-model entity class features which are especially difficult for implementation.

The set of meta-models presented in Figures 1–6 is limited to chains and trees only. All the meta-models shown in these figures are elaborated to test the possibility of building any meta-model in the form of a graph and should be interpreted as the proof of the concept. Therefore, the ability of constructing meta-models from meta-model building blocks is tested through the meta-models presented below. This approach solves the typical problem also known from modeling tools – how to test the correctness of the potentially infinite number of models. However, the problem known from modeling discipline is moved here to meta-modeling field.

Some notational conventions should be explained here:

- green rectangles – class under test,
- red associations – responsibility under test,
- stereotypes in red – type of a responsibility (the relation class),
- stereotypes in black – logical exclusions of a particular types of responsibilities, which are necessary to test all cases,
- j, k, l postfixes in class names – particular domain classes that fulfill the following logical condition: $i \neq j$ and $i \neq k$ and $j \neq k$,
- m, n postfixes in stereotypes – particular responsibilities that fulfill the following logical condition: $m \neq n$,

The particular classes: R for root class of the meta-model and D_i for domain classes are always tested in combination with their responsibilities – this is why associations going out from the green rectangles are always red according to the convention rules.

The first group of Figures 1–4 shows meta-models that share the same responsibilities between classes involved in testing process. Figure 1 presents a meta-model for testing the correctness of enrichment of root meta-model class (R) by the same responsibilities (R_m).

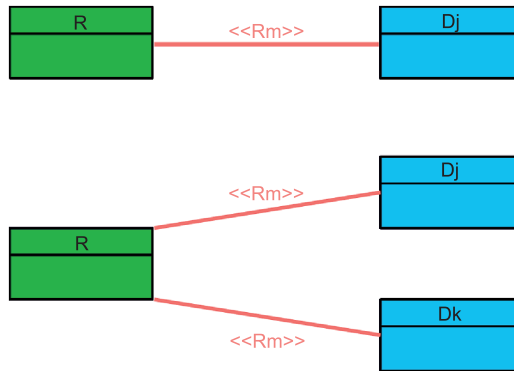


Fig. 1. The same responsibility (R_m) originated in the Root (R) meta-model element

Enrichment of meta-model domain classes (D_j) in the same responsibilities (R_m) is presented in Fig. 2.

Figure 3 is focused on the meta-model which is dedicated to testing the transitiveness of the enrichment of both the root meta-model class (R) and the domain meta-model class (D_j) in the same responsibilities (R_m).

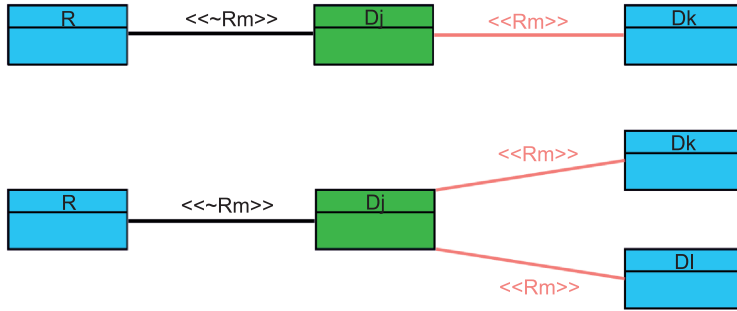


Fig. 2. The same responsibility (Rm) originated in the Domain (Dj) meta-model element



Fig. 3. The same responsibility (Rm) originated in both Root (R) and Domain (Dj) meta-model elements

Similarly, Fig. 4 presents a meta-model used to test the correctness of the transitive enrichment of the two different meta-model domain classes (Dj, Dk) in the same responsibilities (Rm).



Fig. 4. The same responsibility (Rm) originated in different Domain meta-model elements (Dj, Dk)

The Figs. 5–6, on the other hand, contain meta-models that test the correctness of combining different responsibilities injected into same classes. The meta-model root class (R) is enriched in different responsibilities (Rm, Rn) in Fig. 5.

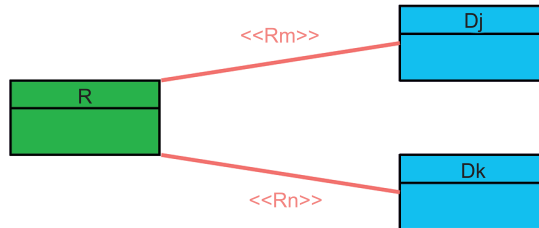


Fig. 5. Different responsibilities (Rm, Rn) originated in the Root (R) meta-model element

Figure 6 presents enrichment of the meta-model domain (Dj) class in different responsibilities (Rm, Rn).

All meta-models presented above were tested by unit tests of each class. The meta-models were also tested by scenario tests. Each scenario test has two parts: one for the

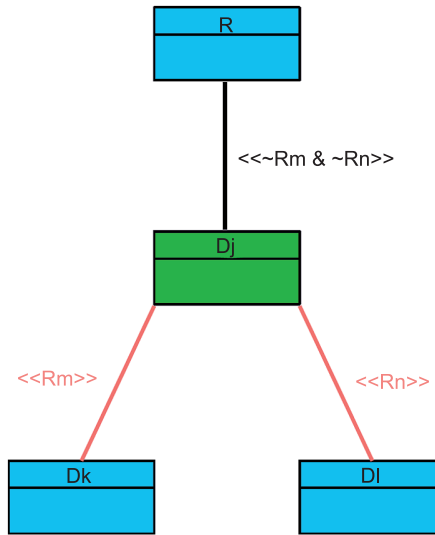


Fig. 6. Different responsibilities (Rm, Rn) originated in the Domain (Dj) meta-model element

model construction and one for the model traversal. Several models (meta-model instances) were created to execute such a particular scenario test.

The framework passed all tests explained above.

4.2. Sample meta-model testing

In this subsection, the sample test scenario is presented for the meta-model shown in Figure 5. However, first the Spring and AspectJ application-context file which constitutes the definition of the modeling language (meta-model) is shown in Fig. 7.

The test scenario for constructing a model which is an instance of the sample meta-model from Fig. 7 and then for traversing the model are presented in Fig. 8.

Such tests were created for all meta-models which were taken into account during the activity of tests strategy planning. They confirmed the correctness of the approach and feasibility of the framework implementation based on the concept presented in the paper.

```

<beans xmlns:aop="http://www.springframework.org/schema/aop"
xmlns="http://www.springframework.org/schema/beans"
xmlns:context="http://www.springframework.org/schema/context"
xmlns:xsi="http://www.w3.org/2001/XMLSchema-instance"
xmlns:int-security="http://www.springframework.org/schema/integration/security"
xmlns:sec="http://www.springframework.org/schema/security"
xsi:schemaLocation="http://www.springframework.org/schema/aop
http://www.springframework.org/schema/aop/spring-aop-3.0.xsd
http://www.springframework.org/schema/security
http://www.springframework.org/schema/security/spring-security-3.0.xsd
http://www.springframework.org/schema/beans
http://www.springframework.org/schema/beans/spring-beans-3.0.xsd
http://www.springframework.org/schema/tx
http://www.springframework.org/schema/tx/spring-tx-3.0.xsd
http://www.springframework.org/schema/context
http://www.springframework.org/schema/context/spring-context.xsd">

    <import resource="../metamodel-core-context.xml"/>
    <import resource="../appContextNamedElement.xml"/>

    <!-- API -->
    <bean class="com.componentcreator.metamodel.coremetamodel.apidynamic.APIdynamic"
        id="coreApiDynamic" scope="singleton"/>

    <!-- Root -->
    <bean class="com.componentcreator.metamodel.coremetamodel.root.R" id="root"
        scope="singleton"/>

    <!-- Domain -->
    <bean class="com.componentcreator.metamodel.coremetamodel.domainsimpl.Dj" id="class"
        scope="prototype"/>
    <bean class="com.componentcreator.metamodel.coremetamodel.domainsimpl.Dk" id="role"
        scope="prototype"/>

    <!-- Responsibility implementations -->
    <bean class="com.componentcreator.metamodel.coremetamodel.responsibilitiesimpl.Rm"
        id="rmImplForR">
        <constructor-arg>
            <value>
                com.componentcreator.metamodel.coremetamodel.domainsimpl.Dj
            </value>
        </constructor-arg>
    </bean>

    <bean class="com.componentcreator.metamodel.coremetamodel.responsibilitiesimpl.Rn"
        id="rnImplForR">
        <constructor-arg>
            <value>
                com.componentcreator.metamodel.coremetamodel.domainsimpl.Dk
            </value>
        </constructor-arg>
    </bean>

    <!-- Responsibility injections -->
    <aop:config>
        <aop:aspect id="relationship" ref="relationshipAspect">
            <aop:declare-parents
                types-matching="com.componentcreator.metamodel.coremetamodel.root.R"
                implement-interface="com.componentcreator.metamodel.coremetamodel.responsibilities.IRm"
                delegate-ref="rmImplForR"/>

            <aop:declare-parents
                types-matching="com.componentcreator.metamodel.coremetamodel.root.R"
                implement-interface="com.componentcreator.metamodel.coremetamodel.responsibilities.IRn"
                delegate-ref="rnImplForR"/>
        </aop:aspect>
    </aop:config>
</beans>

```

Fig. 7. Definition of the sample meta-model in the form of Spring and AspectJ application context


```

@Test
public void test() throws Throwable {

    // obtain initiated interface object
    ICoreMetamodelAPIdynamic ifc =
        Configuration.getCoreMetamodelAPIdynamic(APP_CFG_FNAME);

    // get access to meta-model root singleton
    RootMetamodelCore root = ifc.getRoot();

    // create meta-model domain classes and add them to the root
    Dj dj1 = (Dj) ifc.getDomain(Dj.class, "dj1");
    ((Name)cls1).setName("dj 1");
    Dj dj2 = (Dj) ifc.getDomain(Dj.class, "dj2");
    ((Name)cls2).setName("dj 2");
    Dj dj3 = (Dj) ifc.getDomain(Dj.class, "dj3");
    ((Name)cls3).setName("dj 3");

    BaseMetamodelCoreProxy rootRmAccessorProxy =
        ifc.getAccessor(RootMetamodelCore.class, IRm.class);

    rootRmAccessorProxy.add(root, dj1);
    rootRmAccessorProxy.add(root, dj2);
    rootRmAccessorProxy.add(root, dj3);

    Dk dk = (Dk) ifc.getDomain(Dk.class, "dk");
    ((Name)att).setName("dk 0");

    BaseMetamodelCoreProxy rootRnAccessorProxy =
        ifc.getAccessor(RootMetamodelCore.class, IRn.class);

    rootRnAccessorProxy.add(root, dk);

    System.out.println("Djs count = "+rootRmAccessorProxy.count(root));
    for(int i=0; i<rootRmAccessorProxy.count(root); i++){
        Dj dj = (Dj) rootRmAccessorProxy.get(root, i);
        System.out.println("Dj's name = "+((Name) dj).getName());
    }
    System.out.println();

    Dk dk = (Dk) rootRnAccessorProxy.get(root);
    System.out.println("Dk's name = "+((Name) dk).getName());

    ifc.close();
}

```

Fig. 8. Framework implementation scenario test for the sample model created for a particular meta-model

5. Conclusions

In this paper, the materialization of an idea based on the CDMM-P paradigm was presented. It was shown that the CDMM-F framework being the implementation of the CDMM-P can be realized in the form of a self-organizing software system the structure of which is defined by the application context of this system. The set of risks specific for such an approach was identified and addressed by test cases. The CDMM-F was implemented and tested against these risks. Thus, the proof of the concept for both CDMM-P and CDMM-F was made and presented in the paper.

There is a lot of possible investigative directions that are worthy of taking in future. However, the most important aim at the moment is to elaborate several API versions for the framework. Two such APIs have been defined and tested so far. These require information about the meta-model structure for the purpose of navigation through the model from the test

or framework's client source code. In the next step, the reflective API should be elaborated and tested. As the result of this step the client's source code should become independent of the meta-model structure.

References

- [1] Kalnins A., Vilitis O., Celms E., Kalnina E., Sostaks A., Barzdins J., *Building Tools by Model Transformations in Eclipse*, Proceedings of DSM 2007 workshop of OOPSLA 2007, Montreal, Canada, Jyväskylä University Printing House, p. 194-207.
- [2] Kern H., Kühne S., *Model Interchange between ARIS and Eclipse EMF*, 7th ooPSLA Workshop on Domain-Specific Modeling, Momtreal, Canada, Oktober 2007.
- [3] Kiczales G., Lamping J., Mehdhekar A., Maeda C., Lopes C.V., Loingtier J., Irwin J., *Aspect-Oriented Programming*, Proceedings of the European Conference on Object-Oriented Programming (ECOOP), Springer-Verlag LNCS 1241, June 1997.
- [4] Kleppe A.G., Warmer J., Bast W., *MDA Explained: The Model Driven Architecture: Practice and Promise*, Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA 2003.
- [5] [Krahn H., Rumpe B., Völkel S., *Efficient Editor Generation for Compositional DSLs in Eclipse*, Proceedings of the 7th OOPSLA Workshop on Domain-Specific Modeling DSM' 07, Montreal, Quebec, Canada, Technical Report TR-38, Jyväskylä University, Finland 2007.
- [6] Odersky M., Micheloud S., Mihaylov N., Schinz M., Stenman E., Zenger M., et al., *An overview of the Scala programming language*, EPFL, Lausanne, Switzerland 2004.
- [7] Object Management Group, *Meta Object Facility (MOF) core specification version 2.0*, 2006, <http://www.omg.org/spec/MOF/2.0>.
- [8] Object Management Group, *Unified Modeling Language (UML) superstructure version 2.2*, 2009, <http://www.omg.org/spec/UML/2.2>.
- [9] Selic B., *A systematic approach to domain-specific language design using UML*, Object-Oriented Real-Time Distributed Computing, IEEE International Symposium on, 0:2–9, 2007.
- [10] Silingas D., Vitiutinas R., Armonas A., Nemuraite L., *Domain-specific Modeling Environment Based on UML Profiles*, [in:] *Information Technologies 2009*, Proceedings of the 15th Conference on Information and Software Technologies, IT 2009, Kaunas, Lithuania, April 23–24, Kaunas University of Technology, Technologija, 2009, p. 167-177.
- [11] Sprinkle J., Mernik M., Tolvanen J.-P., Spinellis D., *What Kinds of Nails Need a Domain-Specific Hammer?*, IEEE Software, Guest Editors' Introduction: Domain Specific Modelling, 26(4): 15–18, July/August 2009.
- [12] Zabawa P., Stanuszek M., *Characteristics of Context-Driven Meta-Modeling Paradigm (CDMM-P)*, Technical Transactions, 3-NP/2014, p. 123-134.

PIOTR ZABAWA*, GRZEGORZ FITRZYK**

ECLIPSE MODELING PLUGIN FOR CONTEXT-DRIVEN META-MODELING (CDMM-META-MODELER)

PLUGIN DO MODELOWANIA W ECLIPSE DLA METAMODELOWANIA STEROWANEGO KONTEKSTEM (CDMM-META-MODELER)

Abstract

The novel and crucial point of this paper is the illustration of the application of close ontology based meta-modeling to defining the open ontology based construction of modeling languages (meta-models). This approach is a generalization of Object Management Group (OMG) standards related to Model-Driven Architecture (MDA). The existing modeling tools supporting the traditional approach can be reused when open ontology based meta-modeling tools are implemented. This paper describes Context-Driven Meta-Modeling Meta-Modeler (CDMM-Meta-Modeler) – the Eclipse Plugin that constitutes such an open ontology based meta-modeling tool. The tool constitutes an implementation of the diagramming aspect for the Context-Driven Meta-Modeling Framework (CDMM-F), which is one of several possible implementations of the Context-Driven Meta-Modeling Paradigm (CDMM-P).

Keywords: meta-model, class diagram, modeling paradigm, Eclipse Plugin, OSGi, ontology, visual modeling

Streszczenie

Nowym i kluczowym elementem artykułu jest ilustracja zastosowania metamodelowania opartego o ontologie zamknięte do konstruowania języków modelowania (meta-modeli) opartego o ontologie otwarte. Podejście to jest uogólnieniem standardów Object Management Group (OMG) związanych z Model-Driven Architecture (MDA). Do implementowania narzędzi modelowania opartych o ontologie otwarte można wykorzystać istniejące narzędzia modelowania oparte na tradycyjnym podejściu. Artykuł opisuje Context-Driven Meta-Modeling Meta-Modeler (CDMM-Meta-Modeler) - Plugin do Eclipse'a, który stanowi takie narzędzie do metamodelowania oparte na ontologiach otwartych. Narzędzie to stanowi implementację problemu konstruowania diagramów dla frameworku Context-Driven Meta-Modeling Framework (CDMM-F), który jest jedną z możliwych implementacji paradygmatu Context-Driven Meta-Modeling Paradigm (CDMM-P).

Słowa kluczowe: meta-model, diagram klas, diagram, paradygmat modelowania, Eclipse Plugin, OSGi, ontologia, modelowanie wizualne

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1. Introduction

Contemporary approaches to the creation of modeling languages for software engineering discipline are based on close ontologies – close systems of notions reflected by the static structure of modeling languages represented by meta-models. The characteristic feature of such systems is that each meta-model class is placed in a meta-model graph and this class depends on all other meta-model classes it is related to. As the result, close ontologies are difficult to change, in contrast to open ontologies.

This paper is related to a different approach to meta-modeling which is based on open ontologies named Context-Driven Meta-Modeling (CDMM). In contrast to the traditional approach briefly characterized above, in the CDMM approach, the whole meta-model graph consists of meta-model classes placed in meta-model graph nodes while the relationship classes characteristic for the CDMM approach are located at the graph edges. Moreover, the meta-model classes are unrelated to one another. This results in a very high level of flexibility in defining the meta-model structure as well as in defining the meta-model elements (entity classes and relation classes).

An important special feature of the CDMM concept is that it is not build on top of standards defined to support ontologies like the Resource Description Framework (RDF) by the World Wide Web Consortium (W3C) or the Web Ontology Language (OWL) by the Object Management Group (OMG). As a consequence, the CDMM approach differs significantly from [1, 11, 3, 4, 7, 8, 9, 10, 6]. In contrast to the ontology research domain, it is defined in the manner known from the meta-modeling of software intensive systems known, for example, from [2]. As a result, the CDMM approach can be compared to such extensively used OMG standards like the Meta-Object Facility (MOF) [13], the Unified Modeling Language (UML) [14] and Model-Driven Architecture (MDA) [12]. Thus, the CDMM constitutes the generalization of all the above-mentioned MDA-related standards.

This generalization was achieved through the introduction of a new modeling paradigm named the Context-Driven Meta-Modeling Paradigm (CDMM-P) presented in [15]. This paradigm makes it possible to replace traditional interrelated compact systems of notions both specified and implemented (references between classes inside classes to represent class relationships) via class diagrams by meta-models created during run-time from decoupled meta-model entity classes and meta-model relationship classes. The CDMM approach replaces fixed data structures with dynamically created data structures.

The CDMM-P has its implementation in the form of the Context-Driven Meta-Modeling Framework (CDMM-F) introduced in [16] and explored in [17] from the special perspective of the role the context takes in this framework implementation. The CDMM-F plays the role of proof of the concept for the CDMM-P idea. The CDMM-F framework is implemented in Java technologies – Java SE enriched by Spring and AspectJ. An illustrative case-study for the application of the CDMM-F framework is presented in [18], where the modeling language is defined from scratch just to illustrate the features of the CDMM-F. In this paper, the same meta-model is created from the CDMM-Meta-Modeler especially to illustrate the difference between both approaches.

This paper is related to the CDMM-F framework and the system presented here constitutes an extension for the CDMM-F. The first version of the CDMM-Meta-Modeler

was presented in [5]. The software system introduced in the paper is dedicated to creating static UML models of modeling languages (meta-models) via class diagrams. A meta-model created in this way is then used to create CDMM-F instance just for this meta-model (the model). The model can be then accessed via the API of the CDMM-F. In this way, the concept of MDA-like modeling in the open ontologies is achieved and implemented. Moreover, the paper contains a constructive illustration of the way the commonly known and widely applied close ontology based approach to meta-modeling can be used to define an open ontology based approach to the construction of modeling languages. Thus, the paper constitutes the proof of concept for such a new approach to meta-modeling.

2. Context-Driven Meta-Modeling

Two elements of the Context-Driven Meta-Modeling set of tools that together constitute Context-Driven Meta-Modeling Technology (CDMM-T) are presented below in this section. Firstly, the most fundamental CDMM-F framework characteristic features are shown. Secondly, the role and the concept of visual modeling support for the CDMM-F is presented.

2.1. Framework

The CDMM-F framework is the core element of the whole CDMM-T. It constitutes an implementation of the CDMM-P paradigm. The name of all the above-mentioned CDMM elements originated just from the way that the CDMM-F framework was implemented. The data structure representing the meta-model has the form of the application context XML file, as the CDMM-F is implemented in Java, Spring and AspectJ. As a consequence, the meta-model takes the role of the context for the CDMM-F application. The running framework is created automatically from that file and the structure of the running framework reflects the meta-model structure. The XML file mentioned above has a relatively complex format and contents. It could be created from any text editor, but this approach is error-prone – this is why the visual modeling tool is desirable to design the meta-model which the application context file can be generated from by this tool.

2.2. Visual Meta-Modeling

The Visual Meta-Modeling feature was added to the CDMM-F to simplify the meta-model designers' job. This simplification was achieved by making the graphical tool available for meta-model graph creation and by minimizing the number and complexity of the solution's user tasks. The graph is built from two kinds of elements:

- classes that are associated with graph nodes – meta-model entity classes,
- classes that are associated with graph edges – meta-model entity relationship classes.

Both groups of classes must be predefined with the same tool. For the association of classes to the meta-model graph, the UML Profiles, and specifically UML stereotypes are applied. The association is via the stereotype name which is generated from the name of the class that is intended to be associated to the node or to the edge of the graph. The process of stereotype generating is one of the functions of the visual meta-modeling tool.

The key concept of the tool is the application of the functionalities of close ontology-based modeling tools for constructing the simple tool that makes it possible to define open ontology-based modeling languages.

3. CDMM-Meta-Modeler Features

According to the concept of visual meta-modeling presented in section 2, and according to the characteristics of CDMM-F presented in [16, 17], the most important features of the CDMM-Meta-Modeler can be specified. They are shown below in this section.

The most desired features of the CDMM-Meta-Modeler at the current stage of research and implementation work are as follows:

- **reuse of existing UML modeling (not diagramming) components,**
- **reuse of the mechanism of UML Profile modeling (not diagramming) offered by available components,**
- **reuse of the mechanism of extending UML modeling components via UML Profiles (the two above-mentioned components must be compatible),**
- **reuse of existing diagramming framework,**
- **potential reuse of existing modeling components for additional tasks** like generating source code from the model, serialization of the meta-model via the standard UML XMI format for interoperability purposes,
- **potential integration of the CDMM-Meta-Modeler with existing commercial modeling products (all features listed above support this feature),**
- **the ability to graphically model (diagramming) limited UML models containing:**
 - **stereotypes in UML Profiles,**
 - **classes with stereotypes for classes introduced in UML Profiles,**
 - **associative relationships (composition, aggregation, association) with stereotypes for relationships introduced in UML Profiles,**
 - **dependency relationships with stereotypes for dependency introduced in UML Profiles,**
 - **packages with their hierarchies,**
- the ability to define meta-model entity classes in separate package,
- the ability to define meta-model relation classes in separate package,
- **the ability to customize predefined CDMM-F framework meta-model classes contained in the separate package,**
- the mechanism of the reuse of the predefined CDMM-F framework meta-model relation class in the user-defined meta-models and vice versa,
- the ability to round-trip engineer the UML Profile for stereotyping entity and relationship meta-model classes,
- **the mechanism of drag-and-drop meta-model entity classes to the meta-model diagram,**
- **the mechanism of introducing the UML Profile stereotypes for entity meta-model classes to these classes on the meta-model diagram,**

- **the mechanism of joining meta-model entity classes on the meta-model diagram via associative and/or dependency relationships (creation of the meta-model graph composed of the user-defined entity and relationship meta-model classes),**
- **the mechanism of introducing UML Profile stereotypes for meta-model relations to the relationships put on the meta-model diagram,**
- the ability to load/save the meta-model through the XMI file,
- **the ability to export the meta-model to the XML file for the CDMM-F,**
- **the ability to generate the application context XML for the CDMM-F** and to send this application context to the CDMM-F Plugin in the form of the string.

The most important functionalities from the list shown above were already implemented in the presented CDMM-Meta-Modeler (in bold) while the remaining functionalities are under development. A large number of both functional and non-functional requirements was formally specified according to the RUP's FURPS+ classification and known good practices; however, these are not contained in this paper due to size limitations. However, both functional and non-functional tests discussed shortly in section 5 were implemented according to the requirements correctly specified.

4. CDMM-Meta-Modeler Tool Implementation

The visual modeling concept was materialized in the form of the CDMM-Meta-Modeler tool. The existing modeling tools can be used for the implementation of the CDMM-Meta-Modeler as the concept of the visual modeling assumes that close ontology based modeling can be applied to construct the open ontology based meta-modeling tool. The right source of such close ontology based modeling tools is the Eclipse framework with its Plugins – this is why the CDMM-Meta-Modeler tool was also implemented in the form of the Eclipse Plugin.

The implementation of the Eclipse Plugin makes it possible to achieve functionalities sufficient for the effective creation of open ontology based modeling languages with low expense and independently from commercial modeling software providers. At the same time, the approach typical for meta-modeling can be used in place of the approach characteristic for design and specification of ontologies which is based on RDF or OWL standards.

4.1. Technologies

As previously mentioned, the Eclipse framework was chosen for CDMM-Meta-Modeler tool implementation. The tool itself consists of several Eclipse Plugins. The motivation for such a technological choice is as follows:

- reduction of specialized integrated development environments (IDEs) required by the CDMM-T user to the one based on Eclipse,
- extensibility of the CDMM-T implementation into newly required functionalities via the installation of new Plugins,
- operating system independence of the CDMM-Meta-Modeler tool resulting from Eclipse IDE operation system independence,

- reuse of the UML2 SDK Eclipse Plugin for the UML model of the modeling language storage and for its conformity to the OMG UML standard,
- natural integration with CDMM-F which was already implemented in Java; in fact, it was implemented in Java because all contemporary modeling tools are implemented just in Java, including the UML2 SDK.

The Eclipse framework itself is implemented in Java and is based on Equinox – the Eclipse implementation of Open Services Gateway initiative (OSGi).

Graphical libraries the Eclipse is based on are SWT and JFace. They offer application rendering which is native for Eclipse platform. This is why, in order to unify the presentation style of all platforms, the JavaFX library, the successor of Swing, was applied.

All these technologies are open source which helped to limit cost of the CDMM-Meta-Modeler tool significantly.

The short characteristics of the technologies mentioned above and remaining ones is presented below. This characteristics constitutes motivation for choosing just these technologies.

- Eclipse RCP – This Eclipse Rich Client Platform facilitates the implementation of Rich Client applications based on the dynamic Plugin model via the introduction of a set of dedicated Plugins to the Eclipse Platform.
- Equinox – This is an OSGi implementation dedicated to Eclipse. It helps to manage Plugins and offers extensions that can extend the functionality of other Plugins.
- UML2 SDK – This is the Eclipse PlugIn that includes the implementation of OMG UML version 2.x. The main goals of this component are the provisions of:
 - useful implementation of UML for the support of modeling tool development,
 - common XMI schema for the facilitation of the exchange of semantic models,
 - test cases for the verification of the specification correctness,
 - validation rules as the medium for defining and executing conformance levels.
- SWT – graphical library for Java as an alternative for AWT/Swing libraries. This library has been developed by the Eclipse Foundation. The main idea of this library is to provide native system aspects such as key shortcuts, reactions to mouse behavior etc. as well as the creation of a native application view – this means that the view is unique for each platform.
- JFace – This is one of the Eclipse Foundation's projects which provides graphical components based on SWT. The main goal of the library is to provide such components that are frequently used and, at the same time, are difficult to implement in SWT. As the JFace components constitute an extension of SWT components, they can be combined with SWT.
- JavaFX – This is graphical library which is a successor to Swing. It offers a wide spectrum of components together with the mechanism of their simple customization via CSS files. Moreover, it allows the nesting of its components into other graphical libraries like Swing or SWT. The possibility of nesting JavaFX in SWT allows the use of JavaFX in Eclipse. As a consequence, this allows creating advanced components through the combination of JavaFX, JFace and SWT.
- SWTBot – This is the Java library that allows the implementation of a user interface and/or functional tests for SWT and Eclipse. SWTBot integrates with Tycho very well.

As a result, these tests can be executed by Maven. SWTBot works on all platforms available for SWT. Both Tycho and Maven are used in the development process for the Plugin.

- JUnit – This is the most popular Java-oriented tool for unit testing support. These tests allow for the continuous testing of the product.
- TestFX – In contrast to SWTBot, this library is designed to support the manual development of automated user interface tests and/or functional tests for Swing and JavaFX.

The Eclipse RPC distribution was chosen as it is dedicated to support Eclipse Plugins development and constitutes the execution environment for the CDMM-Meta-Modeler. It allows the integration of all the above-mentioned technologies. The choice of technology was simple as there is no competing open source environment on the market.

4.2. Design

The most challenging part of the design discipline was to take into account the design context resulting from the technologies applied to the CDMM-Meta-Modeler implementation.

The following technologies impacted on the design discipline of the CDMM-Meta-Modeler project: Eclipse, JavaFX, SWT and UML2 Eclipse Plugin. Some parts of the CDMM-Meta-Modeler design elements and the nature of the impact of reused technologies on JavaFX is visible in Figure 1.

Another part of the CDMM-Meta-Modeler design is shown in the diagram presented in Figure 2. It shows the dependency of the design elements presented there on JavaFX and SWT.

The whole CDMM-Meta-Modeler system was carefully designed during the implementation. It is worth noticing here that both Figure 1 and Figure 2 were prepared after implementation of the presented CDMM-Meta-Modeler in this tool for the presentation purposes only.

5. CDMM-Meta-Modeler Tool testing

The CDMM-Meta-Modeler was carefully tested. The test strategy was quite simple as most risks of the product are concentrated in the diagramming functionality and in the integration of the CDMM-Meta-Modeler with modeling tools and with the third-party software presented in section 4.1.

The correctness of the implementation of the CDMM-Meta-Modeler classes was verified by extensive unit testing. The JUnit tool was used to support the process of implementation and the execution of these tests. The testing process was included in Maven's lifecycle.

Another group of tests addressed the verification of the CDMM-Meta-Modeler functionalities constitute functional tests. Some of the tests from this group were implemented with SWTBot and TestFX tools – they were executed automatically in the CDMM-Meta-Modeler development process. However, some of the tests from this group remained manual. The criterion that helped to decide which test should have which form was based on the importance of the tests. The most important tests that covered more functionality had the form of automated tests and some less important and smaller

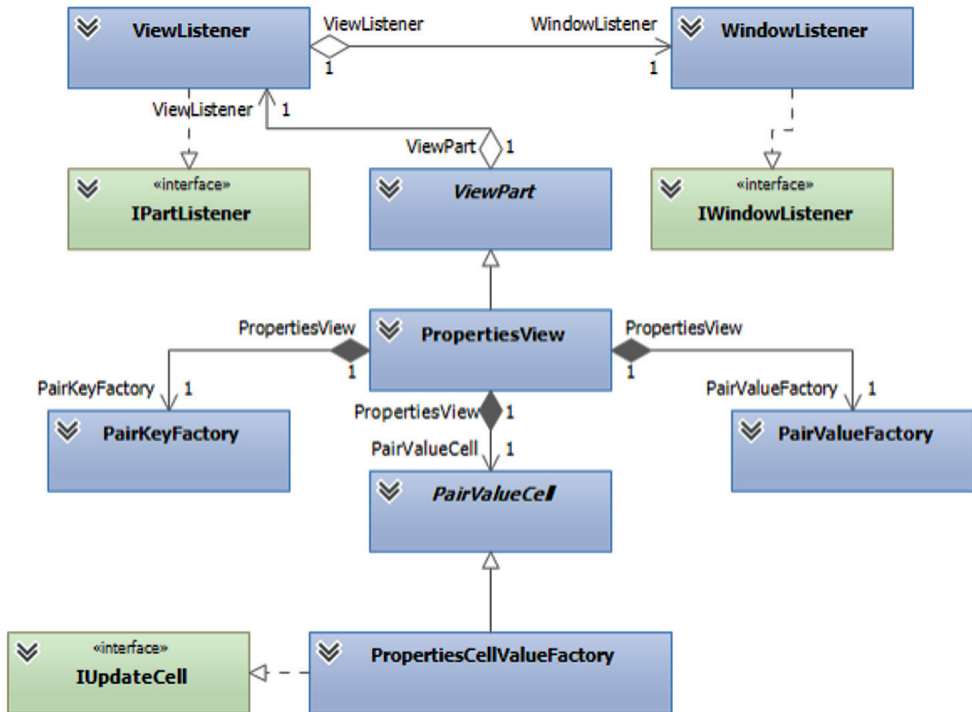


Fig. 2. Class diagram for the 'Properties View' class

```

package com.componentcreator.modeler.metamodel;

import static org.junit.Assert.*;
import org.junit.AfterClass;
import org.junit.BeforeClass;
import org.junit.Test;

public class NewProjectTest {
    private static SWTWorkbenchBot bot;
    @BeforeClass
    public static void beforeClass() throws Exception {
        bot = new SWTWorkbenchBot();
        bot.viewByTitle("Welcome").close();
    }
    @Test
    public void canCreateAnewJavaProject() throws Exception {
        bot.menu("File").menu("New").menu("Project...").click();
        SWTBotShell shell = bot.shell("New Project");
        shell.activate();
        bot.tree().expandNode("Dynamic Meta-Modeling")
            .select("Dynamic Meta-Modeling Modeler Project");
        bot.button("Next >").click();
        bot.textWithLabel("Project name:").setText("TestProject");
        bot.button("Finish").click();
        bot.waitForConditions().shellCloses(shell);
        assertTrue( ResourcesPlugin.getWorkspace().getRoot()
            .getProject("TestProject").exists() );
    }
    @AfterClass
    public static void sleep() {
        bot.sleep(2000);
    }
}

```

Fig. 3. Sample CDMM-F functional SWTBot and JavaFX dependent functional test case

6. Conclusions

The main goal of the research presented in this paper was to show that it is possible to implement a software system which may be used to define open ontology based meta-models in CDMM-P for CDMM-F. This software system is in the form of the Eclipse Plugin named the CDMM-Meta-Modeler which constitutes the physical form of the research result. Moreover, it was shown that such a system can be built using available tools originally designed to support the close ontology based approach to meta-modeling. Specifically, the application of the UML2 modeling Plugin and the UML Profile concept were reused in the CDMM-Meta-Modeler to express open ontology meta-models.

The CDMM-Meta-Modeler tool was carefully designed and tested. In future research, a detailed case study illustrating how the CDMM-Meta-Modeler can be used will be presented. This presentation will be done for a meta-model which is quite simple but expressive enough and applicable to commercial product software development processes.

Some useful functionalities marked in section 3 are still under development – these are planned to be implemented soon in order to make the whole vision of the CDMM-Meta-Modeler presented in the paper workable.

References

- [1] Aßmann U., Zschaler S., Wagner G., *Ontologies, meta-models, and the modeldriven paradigm*, [in:] C. Calero, F. Ruiz, M. Piattini (Eds.), *Ontologies for Software Engineering and Software Technology*, Springer, 2006, p. 249-273.
- [2] Booch G., Rumbaugh J., Jacobson I., *The Unified Modeling Language User Guide*, Addison-Wesley, 2005.
- [3] Djurić D., Devedžić V., *Magic Potion: Incorporating new development paradigms through meta-programming*, IEEE Softw. 27 (5), Sep./Oct. 2010, p. 38-44.
- [4] Djurić D., Jovanović J., Devedžić V., Šendelj R., *Modeling ontologies as executable domain specific languages*, presented at the 3rd Indian Software Eng. Conf., 2010.
- [5] Fitrzyk G., *D-MMF Modeling Tool Based on Eclipse RCP*, MSc. thesis, Cracow University of Technology, Kraków 2014.
- [6] Gallardo J., Molina A., Bravo C., Redondo M., Collazos C., *An ontological conceptualization approach for awareness in domain-independent collaborative modeling systems: Application to a model-driven development method*, Expert Systems with Applications 38, 2011, p. 1099-1118.
- [7] Gašević D., Djurić D., Devedžić V., *Model Driven Engineering and Ontology Development*, Springer-Verlag, 2009.
- [8] Gašević D., Kaviani K., Milanović M., *Ontologies, software engineering*, [in:] *Handbook on Ontologies*. Springer-Verlag, 2009.
- [9] Guizzardi G., *Ontological foundations for structural conceptual models*, Telematica Instituut Fundamental Research Series 15, 2005.
- [10] Guizzardi G., *On ontology, ontologies, conceptualizations, modeling languages and (meta) models*, [in:] *Frontiers in Artificial Intelligence and Applications*, Vol. 155, Conference on Databases and Information Systems IV, IOS Press, Amsterdam, selected Papers from the Seventh International Baltic Conference DB and IS 2006, pp. 18-39.
- [11] Holanda O., Isotani S., Bittencourt I., Elias E., Tenório T., *Joint: Java Ontology Integrated Toolkit*, Expert Systems with Applications 40, 2013, p. 6469-6477.

- [12] Kleppe A. G., Warmer J., Bast W., *MDA Explained: The Model Driven Architecture: Practice and Promise*, Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA 2003.
- [13] Object Management Group, *Meta Object Facility (MOF) core specification version 2.0*, 2006, <http://www.omg.org/spec/MOF/2.0>.
- [14] Object Management Group, *Unified Modeling Language (UML) superstructure version 2.2*, 2009, <http://www.omg.org/spec/UML/2.2>.
- [15] Zabawa P., Stanuszek M., *Characteristics of Context-Driven Meta-Modeling Paradigm (CDMM-P)*, Technical Transactions, 3-NP/2014, p. 123-134.
- [16] Zabawa P., *Context-Driven Meta-Modeling Framework (CDMM-F) – Internal Structure*, 2015 (submitted for publication).
- [17] Zabawa P., *Context-Driven Meta-Modeling Framework (CDMM-F) – Context Role*, 2015 (accepted for publication).
- [18] Zabawa P., *Context-Driven Meta-Modeling (CDMM) – Simple Horizontal Case-Study*, 2015 (submitted for publication).

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