

PIOTR KOT*

PEAK SET ON THE UNIT DISC

ZBIÓR SZCZYTOWY DLA DYSKU JEDNOSTKOWEGO

Abstract

Abstract: We show that any compact subset K in the boundary of the unit disc D with a zero measure is a peak set for $A(D)$.

Keywords:

Streszczenie

Pokażemy, że dowolny podzbiór zwarty K miary zero w brzegu dysku jednostkowego jest zbiorem szczytowym dla $A(D)$.

Słowa kluczowe:

DOI: 10.4467/2353737XCT.16.141.5752

* Piotr Kot (pkot@pk.edu.pl), Institute of Mathematics, Faculty of Physic, Mathematics and Computer Sciences, Cracow University of Technology.

1. Schwarz integral

The goal of this paper is to consider some properties of one-dimensional holomorphic functions in the unit disc. We focus our attention on such boundary properties of these functions which imply their uniqueness. In this aspect Luzin-Privalov theorem [4–6] seems to be crucial. This theorem refers to a meromorphic function $f(z)$ of the complex variable z in a simply-connected domain D with rectifiable boundary Γ . If $f(z)$ takes angular boundary values zero on a set $E \subset \Gamma$ of positive Lebesgue measure on Γ , then $f(z) = 0$ in D . There is no function meromorphic in D that has infinite angular boundary values on a set $E \subset \Gamma$ of positive measure.

We are going to construct some examples of a holomorphic non-constant function f for a given E set of measure zero with $f = 1$ on E .

It will turn out that this E set is a peak set for a proper algebra of holomorphic functions.

We say that a compact set K is a peak set for $A(D)$ if there exists $f \in A(D)$ such that $|f| < 1$ on $\bar{D} \setminus K$ and $f = 1$ on K . Stensönes Henriksen has proved [2] that every strictly pseudoconvex domain with C^∞ boundary in C^d has a peak set with a Hausdorff dimension $2d - 1$.

In this paper we give an alternative, even stronger construction for the unit disc. In the context of the Luzin-Privalov theorem we give the optimal construction for algebra $A(D)$.

Main tool in our construction is the Schwarz kernel.

Let us consider a natural measure σ on boundary of the unit circle ∂D . For a given u which satisfies a Hölder condition we use Schwarz integral (see [7, 8]):

$$Su(z) := \frac{1}{2\pi i} \int_{\partial D} u(t) \frac{t+z}{t-z} \frac{dt}{t}.$$

We can easily observe that $Su \in O(D)$.

Then the Schwarz integral formula Su defining an analytic function, the boundary values of whose real part coincide with u . Additionally, the real part of Su is a continuous harmonic function on \bar{D} (see [1, The Basic Lemma]).

There exists a harmonic function v on D so that $Su = u + iv$.

However when applying the above integral formula, a very important and more difficult problem arises concerning the existence and the expression of the boundary values of the imaginary part v and of the complete function Su by the given boundary values of the real part u . Still, in some cases we have complete information about v .

If a given function u satisfies a Hölder condition, then the corresponding values of imaginary part v on ∂D are expressed by the Hilbert formula (see [3, 1, pp. 45-49]):

$$v(\phi) = -\frac{1}{2\pi} \int_0^{2\pi} u(t) \cot\left(\frac{t-\phi}{2}\right) dt.$$

The above formula is a singular integral and exists in the Cauchy principal-value sense.

Moreover, if u satisfies a Hölder condition then the values of v exist on all $\phi \in \partial D$ and satisfy the same Hölder condition as u . Now we can recover Su using v in the following way:

$$Su(z) := \frac{1}{2\pi} \int_{\partial\mathbb{D}} v(t) \frac{t+z}{t-z} \frac{dt}{t} + c_1.$$

But now the imaginary part of Su is continuous on $\bar{\mathbb{D}}$, so $Su \in A(\mathbb{D})$ if u satisfies a Hölder condition.

2. Peak sets

Lemma 1. *Assume that K, D are distinct compact sets in $\partial\mathbb{D}$. Then there exists a function $u \in C^\infty(\partial\mathbb{D})$ so that $u = 0$ on D , $u = 1$ on K and $0 \leq u \leq 1$ on $\partial\mathbb{D}$.*

Proof. There exist open arcs $I_i : \{e^{2\pi it} : a_i < t < b_i\}$ such that $K \subset \bigcup_{i=1}^n I_i$ and $\bar{I}_i \cap D = \emptyset$. In fact we can assume that $\bar{I}_i \cap \bar{I}_j = \emptyset$ for $i \neq j$. Now there exist functions $u_i : \partial\mathbb{D} \rightarrow [0, 1] \in C^\infty(\partial\mathbb{D})$ so that $u_i = 1$ on I_i , and $u_i = 0$ on D but with distinct supports. It is enough to define $u = \sum_{k=1}^n u_k$.

Theorem 2. *Let K be a compact subset of $\partial\mathbb{D}$ measure zero ($\sigma(K) = 0$). There exists a function $f \in A(\mathbb{D})$ such that $|f| < 1$ on $\bar{\mathbb{D}} \setminus K$ and $f = 1$ on K .*

Proof. Let us choose $\varepsilon > 0$ and define

$$D_\varepsilon := \{z \in \partial\mathbb{D} : \inf_{w \in K} |z - w| \geq \varepsilon\}$$

There exists $u_\varepsilon \in C^\infty(\partial\mathbb{D})$ such that $0 \leq u_\varepsilon \leq 1$, $u_\varepsilon(z) = 0$ if $z \in D_\varepsilon$ and $u_\varepsilon(z) = 1$ if $z \in K$. In particular $Su_\varepsilon \in A(\mathbb{D})$ and $0 \leq \Re Su_\varepsilon \leq 1$.

Let us choose $z \in \bar{\mathbb{D}} \setminus K$ and define $\delta(z, \varepsilon) := \inf_{w \in \partial\mathbb{D} \setminus D_\varepsilon} |z - w|$. We can estimate

$$|Su_\varepsilon(z)| \leq \left| \frac{1}{2\pi} \int_{\partial\mathbb{D} \setminus D_\varepsilon} \frac{t+z}{t-z} \frac{dt}{t} \right| \leq \frac{\sigma(\partial\mathbb{D} \setminus D_\varepsilon)}{2\pi} \max_{t \in U(\varepsilon)} \left| \frac{t+z}{t-z} \right| \leq \frac{\sigma(\partial\mathbb{D} \setminus D_\varepsilon)}{\delta(z, \varepsilon)}.$$

Let us consider the following compact set:

$$T_n : \{z \in \bar{\mathbb{D}} : \inf_{w \in K} |z - w| \geq 2^{-n} + 2^{-2n}\}$$

There exists $\varepsilon_n \in (0, 2^{-2n})$ such that $\sigma(\partial\mathbb{D} \setminus D_{\varepsilon_n}) < 2^{-2n}$. Now let $g_n := Su_{\varepsilon_n} \in A(\mathbb{D})$.

Obviously $\Re g_n = 1$ on K and $0 \leq \Re g_n \leq 1$.

Moreover if $z \in T_n$ then

$$|g_n(z)| \leq \frac{\sigma(\partial\mathbb{D} \setminus D_{\varepsilon_n})}{\delta(z, \varepsilon_n)} \leq \frac{2^{-2n}}{2^{-n} + 2^{-2n} - 2^{-2n}} = 2^{-n}.$$

Now we are able to define $g := 1 + \sum_{n \in \mathbb{N}} g_n$.

Since $\bigcup_{n \in \mathbb{N}} T_n = \bar{D} \setminus K$ we can observe that $g \in O(D) \cap C(\bar{D} \setminus K)$. As $0 \leq \Re g_n \leq 1$ and $\Re g_n = 1$ on K we have $\lim_{z \rightarrow w} \Re g_n(z) = \infty$ for $w \in K$.

Now we choose $f := \exp\left(-\frac{1}{g}\right)$. Obviously $f \in O(D) \cap C(\bar{D} \setminus K)$.

Since $\Re \frac{1}{g} = \frac{\Re \bar{g}}{|g|^2} = \frac{\Re g}{|g|^2} > 0$ on $\bar{\Omega} \setminus K$ we may easily observe that $0 < |f| < 1$ on $\bar{\Omega} \setminus K$.

Additionally due to $\lim_{z \rightarrow w} \frac{1}{|g(z)|} = 0$ for $w \in K$ we have $f = 1$ on K and $f \in A(\Omega)$.

Example 3. *There exists $K \subset \partial D$, a compact set with Hausdorff dimension equal one which is also a peak set for $A(D)$.*

Let us consider a sequence of closed distinct intervals $I_n := [2^{-2n-1}, 2^{-2n}]$. There exists Cantor set $C_n \subset I_n$ with Hausdorff dimension equal $\frac{n}{n+1}$. Now we define a compact set

$$K := \{1\} \cup \bigcup_{n=1}^{\infty} \{e^{2\pi i t} : t \in C_n\}$$

in ∂D with Hausdorff dimension one and due to Theorem 2 we conclude that K is a peak set for $A(D)$.

References

- [1] Gakhov F.D., *Boundary value problems*, Pergamon, 1966 (Translated from Russian).
- [2] Stensönes Henriksen: *A peak sets of Hausdorff dimension $2n - 1$ for the algebra $A(D)$ in the boundary of a domain D with C^∞ -boundary in C^n* , Math. Ann., 259, 1982, 271-277.
- [3] Hilbert singular integral, B.V. Khvedelidze (originator), Encyclopedia of Mathematics: http://www.encyclopediaofmath.org/index.php?title=Hilbert_singular_integral&oldid=11933 [access: 19.12.2016].
- [4] Luzin-Privalov theorems. Encyclopedia of Mathematics: http://www.encyclopediaofmath.org/index.php?title=Luzin-Privalov_theorems&oldid=27205 [access: 19.12.2016].
- [5] Lusin N.N., Priwaloff I.I., Sur l'unicité et la multiplicité des fonctions analytiques, Ann. Sci. Ecole Norm. Sup. (3), 42, 1925, pp. 143-191.
- [6] Priwalow I.I., *Randeigenschaften analytischer Funktionen*, Deutsch. Verlag Wissenschaft, 1956 (Translated from Russian).
- [7] Schwarz integral. Encyclopedia of Mathematics: http://www.encyclopediaofmath.org/index.php?title=Schwarz_integral&oldid=31192 [access: 19.12.2016].
- [8] Schwarz H.A., *Gesamm. math. Abhandl.*, 2, Springer, 1890.