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ON SOME VOLATILITY REDUCTION OF RETURNS ON SHARES

O REDUKCJI ZMIENNOŚCI STOPY ZWROTU Z AKCJI

Abstract

In this paper we consider derivatives which are binary options of asset-or-nothing type with a payoff function depending on a parameter. The payoff is modelled on the payoff of catastrophe bonds. We examine the influence of the derivative on returns on shares. For this purpose two portfolios are compared: one consisting of stocks and a second additionally containing the derivative. Using the Black-Scholes model we derive an explicit formula for the standard deviation of the returns on the investment portfolios. Numerical examples show that the derivative reduces the volatility of returns on shares. For typical values of stock price volatility we indicate the value of the parameter appearing in the payoff for which the volatility of returns on shares reaches a minimum. All numerical calculations were made with MAPLE.

Keywords: Black-Scholes model, risk-reducing derivatives, MAPLE

Streszczenie

W artykule rozważa się pewien pochodny instrument finansowy którego funkcja wypłaty jest wzorowana na funkcji wypłaty z obligacji katastroficznych. Analizuje się wpływ tego instrumentu na stopę zwrotu z akcji porównując portfel akcji z portfelem zawierającym dodatkowo rozważany instrument pochodny. Stosując model Blacka-Scholesa wyprowadza się dokładny wzór na odchylenie standardowe stóp zwrotu z każdego z tych portfeli. Analizowane przykłady pokazują, że rozważany instrument pochodny redukuje zmienność stóp zwrotu z akcji. Obliczenia do podanych przykładów zostały wykonane przy pomocy programu MAPLE.

Słowa kluczowe: model Blacka-Scholesa, instrument pochodny redukujący ryzyko, MAPLE

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1. Introduction

The subject of this paper is a derivative, considered in [4] as a risk-reducing derivative. The payment of the derivative depends on a parameter. Using Monte Carlo simulations, for each of the typical value of the volatility of stocks a variant of the derivative (a proper parameter in a payoff function) reducing the risk of a large loss by more than 10% on a confidence level of 95% was indicated.

In this paper we examine volatility of rate of return from stocks, when portfolio apart from stocks additionally includes a derivative. We obtain an analytical closed form formula for the volatility expressed as standard deviation of related, discounted percentage of profit from a portfolio. We show that the derivative reduces volatility of rate of return on stocks.

In this paper we use the Black-Scholes model with one risk-free asset and one risky instrument – a stock – regarded as the underlying. We consider the simplest case of the model which is based on the following assumptions: security trading is continuous, there are no riskless arbitrage opportunities, there are no transaction costs and no dividends during the life of a derivative, the risk-free rate of interest and the volatility of an underlying asset are constant. The annualized volatility of the stock, from now on called briefly volatility, is typically between 15% and 60% [6].

2. Model description

Let $\sigma > 0$ be a stock price volatility and r be the risk-free interest rate. We assume the price of the stock follows a geometric Brownian motion

$$S_t = S\exp\left(\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right), \quad t \in [0,T] \tag{1}$$

where S is the stock price at time 0, $W = \{W_t, t \in [0,T]\}$ is a standard Brownian motion under the risk-neutral probability P and T is the expiry date. Let E^P denote the expectation operator under the P measure and let $\{\mathcal{F}_t\}$ be a filtration for Brownian motion W. Let us consider a financial derivative instrument dependent on parameter $a > 0$, with the following payoff function

$$f(S_T) = \begin{cases} S_T & \text{if} \quad S_T \leq aS, \\ 0 & \text{if} \quad S_T > aS. \end{cases} \tag{2}$$

The instrument provides some protection against a decline in the stock price i.e. against the event $S_T \leq aS$ and can be considered as an obligation transferring the risk from the holder of the derivative to the issuer [4]. We will analyse a portfolio composed of one stock and one derivative with payoff function (2). We will calculate the variance of the discounted profit from the portfolio. According to the volatility of the stock we will indicate value of a in the interval $[0,2]$ which minimizes the variance.

3. Volatilities of portfolios

In Black-Scholes model, today's arbitrage price of the derivative instrument expresses as the expected value of its discounted payoff function, taken with respect to the risk-neutral measure P [2]:

$$c = E^P(\exp(-rT)f(S_T)) \tag{3}$$

In [4] the following closed form formula for pricing the derivative was derived

$$c = SN\left(\frac{lna - \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \tag{4}$$

where N denotes the cumulative probability distribution function for a standardized normal distribution. The formula can also be found in [2] and [5]. The today's price of considered stock equals S so the discounted gain from a portfolio is

$$(S_T + f(S_T))\exp(-rT) - (S + c)$$

and the related, discounted percentage of profit from the portfolio equals

$$R = \frac{(S_T + f(S_T))\exp(-rT) - (S + c)}{S + c} \cdot 100\%. \tag{5}$$

To calculate standard deviation of R let us first denote:

Φ – cumulative probability distribution function of σW_T,
ϕ – probability density function of σW_T,
F – cumulative probability distribution function of S_T,
f – probability density function of S_T,

$$k = S\exp\left[\left(r - \frac{1}{2}\sigma^2\right)T\right]. \tag{6}$$

By it follows that $S_T = k\exp(\sigma W_T)$ and consequently

$$f(x) = \frac{1}{x}\varphi\left(\ln\frac{x}{k}\right) \quad \text{for} \quad x > 0$$

and

$$f(x) = 0 \quad \text{for} \quad x \le 0 \tag{7}$$

where

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi T}}\exp\left(-\frac{x^2}{2\sigma^2 T}\right).$$

Hence

$$D^2(R)=\left(\frac{100}{S+c}\right)^2 D^2[(S_T+f(S_T))\exp(-rT)-(S+c)]=$$
$$=\left(\frac{100}{S+c}\right)^2 \exp(-2rT)D^2(S_T+f(S_T)).$$

Since

$$f(S_T)=S_T 1_{\{S_T\le aS\}}$$

it follows that

$$(S_T+f(S_T))^2=S_T^2+3S_T^2 1_{\{S_T\le aS\}}.$$

By (3) we have $E^P(f(S_T))=e^{rT}c$. The process $\exp(-rT)S_t,\ t\ge 0$ is a martingale which implies

$$E^P(S_T)=\exp(rT)S. \tag{8}$$

Hence

$$E^P(S_T+f_T)=\exp(rT)(S+c) \tag{9}$$

and variance of R expresses as follows:

$$D^2(R)=\left(\frac{100}{S+c}\right)^2\left[e^{-2rT}(E^P(S_T^2+3S_T^2 1_{\{S_T\le aS\}})-(S+c)^2)\right] \tag{10}$$

Using (7) we calculate $E^P(S_T^2)$ as $\int_0^\infty x\varphi\left(\ln\left(\frac{x}{k}\right)\right)dx.$

Substituting $\ln\left(\frac{x}{k}\right)=t$ we have

$$E^P(S_T^2)=\int_{-\infty}^{\infty}k^2\exp(2t)\varphi(t)dt=\int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi T}}\exp\left(2t-\frac{t^2}{2\sigma^2 T}\right)dt.$$

But

$$2t-\frac{t^2}{2\sigma^2 T}=2T\sigma^2-\left(\frac{t}{\sigma\sqrt{2T}}-\sigma\sqrt{2T}\right)^2$$

and consequently

$$E^P(S_T^2)=\frac{k^2\exp(2T\sigma^2)}{\sigma\sqrt{2\pi T}}\int_{-\infty}^{\infty}\exp\left(-\left(\frac{t}{\sigma\sqrt{2T}}-\sigma\sqrt{2T}\right)^2\right)dt.$$

Taking into account (6) and substituting $\frac{t}{\sigma\sqrt{2T}} - \sigma\sqrt{2T} = \frac{u}{\sqrt{2}}$ we have

$$E^P(S_T^2) = S^2 \frac{\exp(2r+\sigma^2)T}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}u^2\right) du = S^2 \exp[(2r+\sigma^2)T]. \tag{11}$$

Similarly, we calculate $E^P(S_T^2 1_{(S_T \le aS)})$. Namely, with the change of variables we have

$$E^P(S_T^2 1_{(S_T \le aS)}) = \int_{-\infty}^{aS} x^2 f(x)dx.$$

Using substitutions as above, i.e. $\ln\left(\frac{x}{k}\right) = t$ and $\frac{t}{\sigma\sqrt{2T}} - \sigma\sqrt{2T} = \frac{u}{\sqrt{2}}$ and we obtain

$$\begin{aligned}
E^P(S_T^2 1_{(S_T \le aS)}) &= \int_0^{aS} x\varphi\left(\ln\left(\frac{x}{k}\right)\right)dx \\
&= \int_{-\infty}^{A} k^2 \frac{1}{\sigma\sqrt{2\pi T}} \exp\left(2t - \frac{t^2}{2\sigma^2 T}\right)dt \\
&= S^2 \frac{\exp((2r+\sigma^2)T)}{\sigma\sqrt{2\pi T}} \int_{-\infty}^{A} \exp\left(-\left(\frac{t}{\sigma\sqrt{2T}} - \sigma\sqrt{2T}\right)^2\right)dt \\
&= S^2 \frac{\exp((2r+\sigma^2)T)}{\sqrt{2\pi}} \int_{-\infty}^{B} \exp\left(-\frac{1}{2}u^2\right)du \\
&= S^2 \exp((2r+\sigma^2)T)N(B)
\end{aligned}$$

where

$$A = \ln\left(\frac{aS}{k}\right) \quad \text{and} \quad B = \sqrt{2}\left(\frac{\mathrm{A}}{\sigma\sqrt{2T}} - \sigma\sqrt{2T}\right).$$

Using (6) we obtain

$$E^P(S_T^2 1_{(S_T \le aS)}) = S^2 \exp((2r+\sigma^2)T) N\left(\frac{\ln a - \left(r + \frac{3}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \tag{12}$$

Finally, substituting (4), (11) and (12) into (10) we obtain

$$D^2(R)=10^4\left[\frac{\exp(\sigma^2 T)\left[1+3N\left(\dfrac{\ln a-\left(r+\frac{3}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right]}{\left(1+N\left(\dfrac{lna-\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\right)^2}-1\right] \tag{13}$$

To examine the impact of the derivative, defined by (2), on the rate of return on investment in shares, we are going to compare the above variance with variance of analogous rate of return from a portfolio composed of a stock only. Namely, let S be the today's price of considered stock. Then, the discounted gain from a portfolio is

$$S_T\exp(-rT)-S$$

and the related, discounted percentage of profit from the portfolio equals

$$Z=\frac{S_T\exp(-rT)-S}{S}\cdot 100\%. \tag{14}$$

Using (8) and (11) we obtain

$$D^2Z=10^4(\exp(\sigma^2 T)-1). \tag{15}$$

4. Comparison of volatility of *R* and *Z* with MAPLE

In this section we compare two portfolios, one composed of one stock with value $S = 1$ at time 0 and with one derivative with payoff (2), at price c, given by (4).

The second portfolio is composed of one stock with value $S = 0$ at time 0 only. As in the previous section, R and Z denote the related, discounted percentages of profit from the portfolios, respectively. We calculate standard deviations of R and Z using MAPLE. Let us consider standard deviation of R as a function of parameter a.

In the screenshot presented below, due to the requirements of MAPLE, standard deviation of R is denoted as σR and F denotes the cumulative probability distribution function for a standardized normal distribution:

$$\sigma R := a \to 100 \cdot \mathrm{sqrt}\left(\exp(\sigma^2 T) \cdot \frac{1 + 3 \cdot F\left(\frac{\ln(a) - \left(r + \frac{3}{2}\sigma^2\right) \cdot T}{(\sigma \cdot \mathrm{sqrt}(T))} \right)}{\left(1 + F\left(\frac{\ln(a) - \left(r + \frac{1}{2}\sigma^2\right) \cdot T}{(\sigma \cdot \mathrm{sqrt}(T))} \right)\right)^2} - 1 \right).$$

Let $\sigma_{min}(R)$ denote the minimum of σR, considered as a function of parameter $a \in \left[\frac{1}{10}, 2\right]$ and let a^* be the value of the parameter for which the function takes the minimum value.

We obtain $\sigma_{min}(R)$ and a^* using command of MAPLE:

$$NLPSolve(\sigma R(a),\ a = 1/10..2).$$

In Table 1 one can see dependence of a^* and $\sigma_{min}(R)$ from σ.

For every σ appearing in the table, a^* and $\sigma_{min}(R)$ take the same values, independently of $r \in \{1\%, 2\%, 3\%, 4\%, 5\%, 6\%\}$.

Table 1

σ [%]	10	20	30	40	50	60	70	80	90
a^*	1.36	1.31	1.03	1.02	1.07	1.14	1.25	1.39	1.59
$\sigma_{min}(R)$ [%]	10	17.99	20.9	24.43	31.25	40.03	50.18	61.53	74.16

Standard deviation $\sigma(Z)$ of does not depend on the risk-free interest rate r but it does on stock price volatility σ.

In Table 2, we present values of $\sigma(Z)$ depending on stock price volatility σ. As we can see, the stock price volatility σ and standard deviation $\sigma(Z)$ of Z are approximately equal (both are expressed in percentage):

Table 2

σ [%]	10	20	30	40	50	60	70	80	90
$\sigma(Z)$ [%]	10.025	20.202	30.688	41.655	53.294	65.828	79.518	94.68	111.71

Example

We present a sample screenshot with the calculations for the following parameters:

> X:= *Random Variable* (*Normal*(0, 1)) :
> $F(x)$:= *CDF* (X,x) :
> σ := 0.3 :
> T := 1 :
> σZ := 100·sqrt(exp($\sigma^2 T$) – 1);

30.6878288

still assuming that $\sigma = 0.3$, $T = 1$. As you can see in the screenshot below, standard deviation of R, considered as the function σR on interval $\left[\frac{1}{10}, 2\right]$, achieves minimum equal to 20.9011… for the argument $a = 1.05528…$:

with(*Optimization*):

$$NLPSolve\left(\sigma R(a),\ a = \frac{1}{10}..2\right);$$

[20.9011145934436868, a = 1.05528207323025392]

The same can be seen in a graph of function σR:

$$> plot\left(\sigma R(a),\ a = \frac{1}{10}..2\right)$$

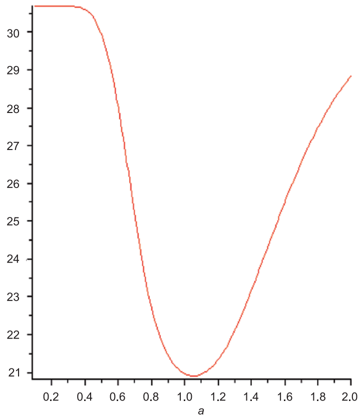

Fig. 1. Dependence of standard deviation of R from parameter a

The following graph allows us to compare the volatilities of return of the considered portfolios:

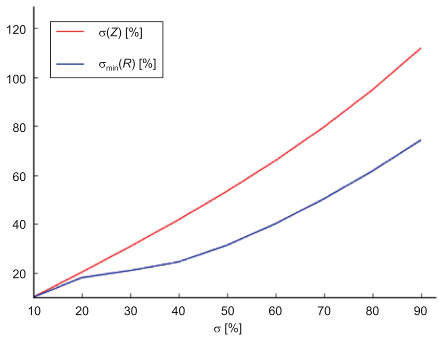

Fig. 2. Volatilities of considered portfolios

5. Conclusions

As shown, σ and $\sigma(Z)$ are approximately equal when $\sigma \leq 30\%$. If $\sigma \leq 30\%$ then $\sigma(Z) > \sigma$ and their difference increases with increasing σ.

Tables 1 and 2 allow us to compare volatilities of Z and R, expressed as standard deviations of Z and R. We see that for every stock price volatility observed in the financial market we can point to such version of considered derivative (with such a parameter a^*) with payoff function (2) that most reduces the volatility of return on the portfolio, thus reducing the risk of investing in stocks.

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