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Homotopy perturbation method for solving fourth – order boundary value problems with additional boundary condition

Homotopijna metoda perturbacyjna zastosowana do zagadnienia brzegowego czwartego rzędu z dodatkowym warunkiem brzegowym

Abstract

This paper presents the homotopy perturbation method for solving linear and non–linear two–point boundary value problems in the form of a fourth–order differential equation and five boundary conditions. Three initial and two final conditions were taken into account. The solution of this problem is possible only when the considered equation includes an unknown parameter. The presented method has been illustrated with a numerical example.

Keywords: boundary value problem, homotopy perturbation method, system of integral equations, system of differential equations

Streszczenie

W artykule przedstawiono homotopijną metodę perturbacyjną zastosowaną do rozwiązywania zarówno liniowego, jak i nieliniowego dwupunktowego zagadnienia brzegowego składającego się z równania różniczkowego czwartego rzędu oraz pięciu warunków brzegowych. Pod uwagę wzięto trzy początkowe i dwa końcowe warunki brzegowe. Rozwiązanie tak postawionego problemu jest możliwe tylko wtedy, gdy rozpatrywane równanie zawiera nieznany parametr. Prezentowaną metodę zilustrowano przykładem obliczeniowym.

Słowa kluczowe: zagadnienie brzegowe, homotopijna metoda perturbacyjna, układ równań całkowych, układ równań różniczkowych

1. Introduction

The boundary value problem (BVP) plays an important role in many fields, e.g. in mathematical modeling, physical and engineering sciences. Most methods of solving these problems concern the standard BVP, where the order of equation and number of boundary conditions are the same. Sometimes we have a non–standard BVP, where the number of boundary conditions is more than the order of a differential equation. In such a situation we can apply e.g. the iterative shooting method (ISM) [1], whose modification was described in [2] or a modification [3] of the variational iteration technique (VIT) [4], which is based on the variational iteration method (VIM) [5, 6].

In this paper, a different alternative method will be presented – the homotopy perturbation method (HPM) [7, 8], modified and applied for solving non–standard higher–order BVP. An example is given to illustrate this method.

2. Basic Concept of the Homotopy Perturbation Method

We will take into consideration the system of the Volterra integral equations, which is written in the matrix form [7, 9]:

$$\mathbf{F}(t)=\mathbf{G}(t)+\lambda\int_a^t \mathbf{K}(t,s)\mathbf{F}(s)ds \tag{1}$$

$$\mathbf{F}(t)=\begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix},\ \mathbf{G}(t)=\begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix},\ \mathbf{K}(t,s)=\begin{bmatrix} k_{11}(t,s) & \cdots & k_{1n}(t,s) \\ \vdots & \ddots & \vdots \\ k_{n1}(t,s) & \cdots & k_{nn}(t,s) \end{bmatrix} \tag{2}$$

To illustrate the basic idea of the HPM [10, 11] and present its application to the system (1), in the first step we consider a general equation:

$$L(u)=0 \tag{3}$$

where L is a differential or integral operator. Next, according to the homotopy perturbation technique, we construct a homotopy operator:

$$H(u,p)=(1-p)F(u)+pL(u)=0,\quad F(u)=L(u)-L(v_0),\quad p\in[0,1] \tag{4}$$

where:

p – an embedding parameter,
$F(u)$ – a functional operator,
v_0 – an initial approximation of (3). Taking into consideration (4), we have:

$$H(u,0)=F(u),\quad H(u,1)=L(u) \tag{5}$$

The process of changing p from zero to unity shows that $H(u, p)$ changes from a starting point $H(v_0, 0)$ to a solution $H(u, 1)$. This is called deformation. The HMP uses the p s an expanding parameter [11] and in the next step we search for the solution of (4) which can be written as a power series in p:

$$u=\sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2 u_2 + \dots , \qquad y=\lim_{p\to 1} u=\sum_{i=0}^{\infty} u_i \tag{6}$$

If $p \to 1$ then (4) becomes the approximate solution of (3). The convergence of series (6) was proved by He [10].

Considering (1) and (6), we have the following system:

$$y_1(t)=\sum_{i=0}^{\infty} p^i z_i , \quad y_2(t)=\sum_{i=0}^{\infty} p^i h_i , \quad y_3(t)=\sum_{i=0}^{\infty} p^i s_i ,\dots \tag{7}$$

By comparing the expressions with the same powers of parameter p, we receive the solution of imposed order.

3. Homotopy Perturbation Method – BVP with additional boundary condition

We consider the non–standard BVP, which consists of the following fourth–order differential equation:

$$u^{(\mathrm{IV})}(x)+f\left(x,u(x),u'(x),u''(x),u'''(x),q_1\right)=0 \tag{8}$$

and five boundary conditions:

$$u(a)=u_a , \;\; u'(a)=u_{1a} , \;\; u''(a)=u_{2a} , \;\; u(b)=u_b , \;\; u'(b)=u_{1b} \tag{9}$$

A solution of equation (8) can fulfill conditions (9) only when this equation contains one unknown parameter q_1. By means of the following transformations:

$$\frac{du}{dx}=z(x), \quad \frac{dz}{dx}=h(x), \quad \frac{dh}{dx}=s(x) \tag{10}$$

we can convert the BVP (8) and (9) to an initial value problem (IVP), which consists of a system of four first – order differential equations:

$$\frac{du}{dx}=z(x), \quad \frac{dz}{dx}=h(x), \quad \frac{dh}{dx}=s(x), \quad \frac{ds}{dx}=f\left(x,u(x),z(x),h(x),s(x),q_1\right) \tag{11}$$

and four initial conditions, which include a subsequent unknown parameter q_2:

$$u(a)=u_a , \;\; z(a)=u_{1a} , \;\; h(a)=u_{2a} , \;\; s(a)=q_2 \tag{12}$$

We can rewrite system (11) as a system of four integral equations:

$$u(x)=u(a)+\int_0^x z(t)dt,\quad z(x)=z(a)+\int_0^x h(t)dt,\quad h(x)=h(a)+\int_0^x s(t)dt$$
$$s(x)=s(a)+\int_0^x f\left(t,u(t),z(t),h(t),s(t),q_1\right)dt \tag{13}$$

If we use (4) and (6) for (13), we will obtain:

$$\begin{cases} u_0+pu_1+p^2u_2+\ldots=u(a)+p\int_0^x\left(z_0+pz_1+p^2z_2+\ldots\right)dt \\ z_0+pz_1+p^2z_2+\ldots=z(a)+p\int_0^x\left(h_0+ph_1+p^2h_2+\ldots\right)dt \\ h_0+ph_1+p^2h_2+\ldots=h(a)+p\int_0^x\left(s_0+ps_1+p^2s_2+\ldots\right)dt \\ s_0+ps_1+p^2s_2+\ldots=s(a)+p\int_0^x f\left(t,u(t),z(t),h(t),s(t),q_1\right)dt \end{cases} \tag{14}$$

Comparing the coefficient of like powers of p, we obtain subsequent approximations of $u(x)$, $z(x)$, $h(x)$, $s(x)$, which include still unknown parameters q_1 and q_2. Using (6) and two boundary conditions (9) at the right end of the domain, we obtain an additional system of two equations:

$$u(b)=u_b\,,\ u'(b)=u_{1b} \tag{15}$$

The solution of system (15) gives the values of parameters q_1, q_2 and $u(x)$– sought solution of equation (8) in terms of convergent series with a required expansion order, which fulfills all conditions (9).

4. Numerical Example

To apply HPM to BVP with an additional boundary condition, an example will be presented as a fourth–order differential equation which contains unknown q_1:

$$u^{(\mathrm{IV})}(x)=u^2(x)-x^{10}+4x^9-4x^8-4x^7+8x^6-4x^4+q_1x-48 \tag{16}$$

and five boundary conditions:

$$u(0)=0,\ u'(0)=0,\ u''(0)=4,\ u(1)=1,\ u'(1)=1 \tag{17}$$

We know ([4]) that for $q_1 = 120$, equation (16) has the following exact solution:

$$u(x)=x^5-2x^4+2x^2 \tag{18}$$

which fulfills all boundary conditions (17). In the first step, using (10), we rewrite the above BVP as a system of four first–order differential equations:

$$\begin{aligned}&\frac{du}{dx}=z(x),\quad \frac{dz}{dx}=h(x),\quad \frac{dh}{dx}=s(x),\\&\frac{ds}{dx}=u(x)-x^{10}+4x^{9}-4x^{8}-4x^{7}+8x^{6}-4x^{4}+q_1x-48\end{aligned}\tag{19}$$

with four initial conditions, which include a second unknown parameter q_2:

$$u(0)=0,\ z(0)=0,\ h(0)=4,\ s(0)=q_2\tag{20}$$

Taking into consideration (14), in the next step we can rewrite the system (19) as a system of four integral equations with the embedding parameter p:

$$\begin{cases}u_0+pu_1+p^2u_2+\ldots=0+p\int_0^x(z_0+pz_1+p^2z_2+\ldots)dt\\ z_0+pz_1+p^2z_2+\ldots=0+p\int_0^x(h_0+ph_1+p^2h_2+\ldots)dt\\ h_0+ph_1+p^2h_2+\ldots=4+p\int_0^x(s_0+ps_1+p^2s_2+\ldots)dt\\ s_0+ps_1+p^2s_2+\ldots=q_2+p\int_0^x[(u_0+pu_1+p^2u_2+\ldots)^2-t^{10}+\\ \qquad\qquad +4t^9-4t^8-4t^7+8t^6-4t^4+q_1t-48]dt\end{cases}\tag{21}$$

The *Maple*™ program with accuracy *Digits* = 20 and order of power series in an embedding parameter n = 20 were used to solve this non–standard problem. Using the boundary conditions at x = 1, we obtain: q_1 = 120.000004947, q_2 = –2.10539 · 10^{-7}, and the series solution is given by:

$$u(x)=2x^2-3.52\cdot10^{-8}x^3-2x^4+x^5-4.65\cdot10^{-11}x^9+\ldots+6.73\cdot10^{-20}x^{38}\tag{22}$$

Table 1 exhibits the exact solutions (18), a comparison between the errors obtained by means of the modified HPM, VIT [3] and ISM [2], presented in this paper and used for the BVP with additional boundary conditions.

Table 1. Error estimates

x	$u_{exact}(x)$	Errors* (HPM)	Errors* (VIT)	Errors* (ISM)
0.0	0.00000	0.0000	0.0000	0.0000
0.1	0.01981	3.4678 10^{-11}	3.4279 10^{-14}	9.8256 10^{-15}
0.2	0.07712	2.6753 10^{-10}	2.6465 10^{-13}	8.0465 10^{-14}
0.3	0.16623	8.4726 10^{-10}	8.3881 10^{-13}	8.4766 10^{-14}
0.4	0.27904	1.8236 10^{-9}	1.7599 10^{-12}	3.4477 10^{-14}

Table 1. cont

0.5	0.40625	$3.0982\ 10^{-9}$	$1.4173\ 10^{-12}$	$1.0776\ 10^{-13}$
0.6	0.53856	$4.3754\ 10^{-9}$	$2.3855\ 10^{-11}$	$1.0341\ 10^{-13}$
0.7	0.66787	$5.1241\ 10^{-9}$	$2.9626\ 10^{-10}$	$4.0618\ 10^{-14}$
0.8	0.78848	$4.6096\ 10^{-9}$	$2.2899\ 10^{-9}$	$3.4080\ 10^{-14}$
0.9	0.89829	$2.3140\ 10^{-9}$	$1.3479\ 10^{-8}$	$5.9746\ 10^{-14}$
1.0	1.00000	$2\ 10^{-19}$	$6.4569\ 10^{-8}$	$1.6930\ 10^{-13}$

*Error = abs (exact solution – series solution (HPM), series solution (VIT) or discreet solution (ISM)).

5. Summary

Taking into consideration methods for the non–standard higher order BVP: the ISM [2], the VIT [3] and HPM described in this article, which can be successfully applied for the BVP with one or more additional conditions, we can conclude that the differential equation must contain unknown parameters, whose number must correspond to the number of excessive boundary conditions. The values of these components can be calculated by applying additional boundary conditions. Taking into consideration the HPM, like the VIT, we obtain a solution in the terms of convergent series. Higher accuracy can be obtained by increasing the expansion order in series solution, but it has impact on time consuming computational work. For example, for $n = 20$, the max error is $5.12 \cdot 10^{-9}$, whereas for $n = 30$ it is reduced to $3.74 \cdot 10^{-12}$. The HPM is easy to implement, powerful and efficient in finding analytical solutions of differential equations.

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