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TIME-AVERAGED VELOCITY PROFILE MODEL OF DRAG REDUCING POLYMER SOLUTIONS IN THE TURBULENT PIPE FLOW

MODEL UŚREDNIONEGO W CZASIE PROFILU PRĘDKOŚCI ROZTWORÓW POLIMEROWYCH WYKAZUJĄCYCH REDUKCJĘ OPORÓW W TURBULENTNYM PRZEPŁYWIE W RURACH

A b s t r a c t

The qualitative, theoretical analysis of macromolecular polymer additive effect on abnormal deformation of the velocity profile of the polymer solution flowing in turbulent motion in smooth pipes of circular cross section is presented. Comparison of the substitutive velocity profiles built in the work shows that for the laminar flow polymer additives do not affect its shape. For the turbulent flow time-average, dimensional velocity profile is always more extended comparing to its Newtonian or purely-viscous equivalent at the same Reynolds numbers. The results suggest that this phenomenon increases with increasing values of the characteristic time constant λ, representing viscoelastic properties of the solution, and with decrease in the diameter of the pipe where the flow is performed.

Keywords: velocity profile, pipe flow, drag reduction

S t r e s z c z e n i e

Przedstawiono jakościową, teoretyczną analizę wpływu dodatków wielkocząsteczkowych polimerów na anormalną deformację profilu prędkości roztworu polimerowego przepływającego ruchem turbulentnym w gładkich rurach o przekroju kołowym. Z porównania zbudowanych modeli zastępczych profili prędkości wynika, że w zakresie laminarnym dodatek polimeru nie wpływa na jego kształt. W zakresie turbulentnym uśredniony w czasie wymiarowy profil prędkości jest zawsze bardziej wydłużony od jego newtonowskiego lub czystolepkiego nienewtonowskiego odpowiednika przy takich samych liczbach Reynoldsa. Wykazano ponadto, że zjawisko to nasila się wraz ze wzrostem wartości charakterystycznej stałej czasowej λ, reprezentującej własności lepkosprężyste roztworu oraz w miarę zmniejszania się średnicy rury, w której realizowany jest przepływ.

Słowa kluczowe: profil prędkości, przepływ w rurze, redukcja oporów

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1. Introduction

One of the characteristic and also less known effects associated with drag reduction phenomenon caused by very small amounts of high-molecular-weight polymer additives is abnormal deformation and elongation of time-averaged velocity profile in flow direction within the turbulent flow range [1, 2, 4, 5].

A full explanation and examination of real structure of such solution, sometimes called "Toms liquid", in turbulent flow by the strictly theoretical considerations is impossible.

The studies in this field are based mainly on the Prandtl's "mixing length" hypothesis [3], introducing merely various modifications on formula defining its relationship on the distance of a fluid element from the wall [2, 4, 6] or expression describing the additional "turbulent viscosity" [5].

In this approach the flow in the pipe is treated as axisymmetric turbulent boundary layer. For its entire cross-section the shear rate is constant $\tau = \tau_w$.

Thus, it boils down to assumption of the gradientless flow, i.e. $dp/dx = 0$ which is obviously not true. Moreover, the disadvantage of proposed in scientific literature modifications of universal velocity profile is fact that derivative of velocity in the tube axis (dv'/dy') is not equal to zero.

Under the Matras hypothesis [7], introduction of high-molecular-weight polymer additives to visco-elastic liquid causes that flow curve of polymer solution obtained from rheometric methods is indeed deformed - due to the occurrence of additional tangential stress during the shear – pseudonewtonian liquid curve.

$$\tau = \eta\left(-\frac{dv}{dr}\right) \tag{1}$$

The fictional pseudo Newtonian liquid with a viscosity $\boldsymbol{\eta}$, which flows in fictional pipe with diameter $\boldsymbol{D}$ with mean velocity $\boldsymbol{v_m}$, satisfies the conditions of the hydrodynamic analogy [7] and can be treated as a Newtonian substitute for the non-Newtonian liquid in pipe flow but following dependencies are valid:

$$\boldsymbol{r} = \left[\frac{2(n+1)}{3n+1}\right]^{\frac{1}{2}}\left(\frac{r}{R}\right)^{\frac{1-n}{2n}} r \tag{2}$$

$$\boldsymbol{\tau} = \left[1+\left(\lambda\frac{8v_m}{D}\right)^n\right]\tau \tag{3}$$

$$\boldsymbol{\eta} = K\left(\frac{3n+1}{4n}\right)^n\left[\frac{2(n+1)}{3n+1}\right]^2\left(\frac{8v_m}{D}\right)^{n-1}\left[1+\left(\lambda\frac{8v_m}{D}\right)^n\right] \tag{4}$$

$$\boldsymbol{R} = \left[\frac{2(n+1)}{3n+1}\right]^{\frac{1}{2}} R \tag{5}$$

$$\boldsymbol{v_m} = \left[\frac{2(n+1)}{3n+1}\right]^{-1} v_m \tag{6}$$

The theoretical analyses carried out in [7] showed that in pseudonewtonian dimensionless quantities system

$$\boldsymbol{c}_f = \frac{2\boldsymbol{R}\boldsymbol{v}_m\rho}{\eta} = \boldsymbol{c}_{fM}\left[1+\left(\lambda\frac{8v_m}{D}\right)^n\right] \tag{7}$$

$$\mathbf{Re} = \frac{\boldsymbol{R}\Delta p}{L\rho\mathbf{v}_\mathrm{m}^2} = \boldsymbol{c}_{fM}\left[1+\left(\lambda\frac{8v_m}{D}\right)^n\right]^{-1} \tag{8}$$

drag reduction of viscoelastic polymer solutions flow in pipes can be described in the laminar flow using the modified Fanning equation:

$$\boldsymbol{c}_f = \frac{64}{\mathbf{Re}} \tag{9}$$

while in the turbulent flow using the modified Blasius equation:

$$\boldsymbol{c}_f = 0.079\mathbf{Re}^{-0.25} \tag{10}$$

In case of the power-law fluid $\lambda = 1$, and therefore:

$$\boldsymbol{c}_f = \boldsymbol{c}_{fM} = \frac{R\Delta p}{L\rho v_m^2}\left[\frac{2(n+1)}{3n+1}\right]^{2.5} \tag{11}$$

It is shown that abnormally elongated velocity profile in range of turbulent flow is caused by viscoelastic properties of liquid, defined in fluid model by the characteristic time constant λ.

2. Time-averahed velocity profile

Consider first a pseudonewtonian model of drag reducing fluid flow in the pipe. Define the following dimensionless variables and expressions:

$$\mathbf{r}' = \frac{\boldsymbol{r}\boldsymbol{v}_m\rho}{\eta} \tag{12}$$

$$\boldsymbol{R}' = \frac{\boldsymbol{R}\boldsymbol{v}_m\rho}{\eta} \tag{13}$$

$$L' = \frac{\boldsymbol{L}\boldsymbol{v}_\mathrm{m}\tilde{\mathrm{n}}}{\mathrm{ç}} \tag{14}$$

$$\Delta\boldsymbol{p}' = \frac{\Delta\boldsymbol{p}}{\rho\boldsymbol{v}_m^2} = \frac{\Delta\boldsymbol{p}_*}{\rho\boldsymbol{v}_m^2} \tag{15}$$

$$\boldsymbol{v}' = \frac{v}{\boldsymbol{v}_m} \tag{16}$$

Taking into account expressions (12)-(16) in equation (1), it can be transformed to following dimensionless form:

$$\tau' = -\frac{d\boldsymbol{v'}}{d\boldsymbol{r'}} \tag{17}$$

In the turbulent range of flow the above equations is no longer valid.

Assume the turbulent, quasi-stationary, pseudonewtonian fluid flow. The time-averaged, dimensionless velocity distribution law – at constant Reynolds number **Re** = const – takes general form:

$$\overline{\boldsymbol{v'}} = f(\boldsymbol{r'}) \tag{18}$$

while the relationship between the dimensionless shear stress and velocity gradient takes general form:

$$\boldsymbol{\tau'} = \varphi\left(-\frac{d\overline{\boldsymbol{v'}}}{d\boldsymbol{r'}}\right) \tag{19}$$

or after the separation of variables:

$$d\overline{\boldsymbol{v'}} = -\varphi^{-1}\left(\boldsymbol{\tau'}\right)d\boldsymbol{r'} \tag{20}$$

Forms of functional operators f in equation (19) and φ in equation (20) depend only on Reynolds number value:

$$\mathbf{Re} = \frac{\boldsymbol{R}'}{2} \tag{21}$$

In order to simplify the notation, the overline of time-averaged quantities on further considerations is neglected.

Considering the balance of forces acting on a isolated, cylindrical fluid element with radius ***r'*** and length dx' ($x' = (x\boldsymbol{v}_m\rho)/\eta$), it follows that in the analysed case of the fully developed, axisymmetric and time-averaged, steady turbulent flow in smooth pipe the total dimensionless shear stress is-as in case of laminar flow – a linear function of the dimensionless radius ***r'***, i.e.:

$$\boldsymbol{\tau'} = \frac{\boldsymbol{r'}\,\Delta\boldsymbol{p'}}{2L'} \tag{22}$$

and for ***r'*** = ***R'***:

$$\boldsymbol{\tau}_w' = \frac{\boldsymbol{R'}\,\Delta\boldsymbol{p'}}{2L'} \tag{23}$$

thus,

$$\boldsymbol{\tau'} = \frac{\boldsymbol{r'}}{\boldsymbol{R'}} \boldsymbol{\tau_w'} \tag{24}$$

Present the dimensionless, averaged flow velocity ($\boldsymbol{v_m}$' = 1) in form of definite integral:

$$1 = \frac{1}{\left(\boldsymbol{R'}\right)^2} \int_0^{(R')^2} \boldsymbol{v'} d\left(\boldsymbol{r'}\right)^2 \tag{25}$$

Assuming no slip of fluid elements at the wall ($\boldsymbol{v'}|_{r'=R'} = 0$) and using dependencies (20) and (24) following equation is obtained after the successive transformations:

$$\left(-\frac{d\boldsymbol{v'}}{d\boldsymbol{r'}}\right)_w = \frac{3s+1}{s} \frac{1}{\boldsymbol{R'}} \tag{26}$$

where:

$$s = \frac{d \ \ln\left(\tau_w'\right)}{d \ \ln\left(\dfrac{1}{\boldsymbol{R'}}\right)} \tag{27}$$

The dimensionless variable *s*, defined by the equations (27) can be interpreted as exponent of curve

$$\tau_w' = C\left(\frac{1}{\boldsymbol{R'}}\right)^s \tag{28}$$

tangential at arbitrary point to curve

$$\tau_w' = \varphi\left(-\frac{d\boldsymbol{v}}{d\boldsymbol{r}}\right)_w \tag{29}$$

Thus in general, the value of dimensionless parameters *s* and *C* depend on the value of expression 1/***R'*** on curve (29), drawn in the wall boundary condition. After transformation of the equation (26) with the respect to 1/***R'*** and further after substitution received expression to equation (28), it is obtained:

$$\tau_w' = C\left(\frac{s}{3s+1}\right)^s \left(-\frac{d\boldsymbol{v'}}{d\boldsymbol{r'}}\right)_w^s \tag{30}$$

Equation (30) defines exponential dependence of dimensionless velocity profile at the wall and is a special case of function (19). A comparison of formulas (24) and (7), and moreover (14) and (8) indicates obvious identities:

$$\boldsymbol{\tau}_{w}' = \frac{\boldsymbol{c}_f}{2} \tag{31}$$

$$\boldsymbol{R}' = \frac{\mathbf{Re}}{2} \tag{32}$$

Considering the above dependencies, equation (31) can be transformed to form:

$$\frac{\boldsymbol{c}_f}{2} = 2C\left(\frac{2}{\mathbf{Re}}\right)^{s} \tag{33}$$

Assume that general relationship between dimensionless shear stress and dimensionless velocity gradient in turbulent flow (19) can be replaced by an exponential dependence, special case of which is equation (30), i.e.:

$$\tau' = C\left(\frac{s}{3s+1}\right)^{s}\left(-\frac{d\boldsymbol{v}'}{d\boldsymbol{r}'}\right)^{s} \tag{34}$$

The velocity profile obtained after integration of the equation (34)

$$\boldsymbol{v}' = \frac{3s+1}{s+1}\left[1-\left(\frac{\boldsymbol{r}'}{\boldsymbol{R}'}\right)\right]^{\frac{s+1}{s}} \tag{35}$$

is a continuous function of the variable $\boldsymbol{r}'$ and has very important common features with the real velocity profile:

– velocity gradient on the wall $(d\boldsymbol{v}'/d\boldsymbol{r}')_{w}$ calculated from the equation (35) is identical to the real velocity gradient (26),
– the velocity gradient in the pipe axis is equal to zero $(d\boldsymbol{v}'/d\boldsymbol{r}')_{r=0} = 0$,
– the average flow velocity, obtained after integration of the equation (35) is equal to the real average flow velocity defined by the equation (25).

Substituting into equation (35) dependences (12), (13), (16), relationships (2), (5) and taking into account fact, that:

$$\frac{v_{\max}}{\boldsymbol{v}_{m}} = \frac{3s+1}{s+1} \tag{36}$$

after transformation it is obtained:

$$\frac{v}{v_{\max}} = \left[1-\left(\frac{r}{R}\right)\right]^{\frac{1+n}{2n}\frac{1+s}{s}} \tag{37}$$

Formula (37) describes the velocity profile of pseudorheostable, power-law fluid for turbulent flow ($s < 1$) and for laminar flow ($s = 1$).

Equations derived above are universal and valid for entire range of Reynolds number. In case of laminar flow $s = 1$ and $C = 4$. Then equation (33) is reduced to familiar form (9). In turbulent range of flow the special case of the equation (33) is modified Blasius equation (10). Parameters s and C take values $s = 0.25$ and $C = 0.033$. In general, with increasing value of Reynolds number decreases the value of the parameter s, tending to zero for $\mathbf{Re} \to \infty$. Equation (34) is also applied to special cases: rheostable power-law fluids and Newtonian fluids.

Prove now, that for turbulent flow, the velocity profile of power-law, drag reducing fluid differs from typical purely-viscous fluid velocity profile (37).

Note that functional relationship (33) presented in dimensionless quantities coordinate system [$\boldsymbol{c}_{fM}$, $\mathbf{Re}_M$] – system characteristic for rheostable, power-law, non-Newtonian fluids in case of turbulent, drag reducing fluidflow – is reduced to family of curves. Indeed, substituting to the equation (33) dependences which define dimensionless quantities $\boldsymbol{c}_f$, $\mathbf{Re}$, i.e. equations (7) and (8), taking into account definitions (11) and (12), after transformations is obtained:

$$\frac{\boldsymbol{c}_{fM}}{2} = C\left(\frac{2}{\mathbf{Re}_M}\right)^s \left\{1+\lambda^n \left[\frac{8K}{\rho}\left(\frac{2n+1}{4n}\right)\left(\frac{2(n+1)}{3n+1}\right)^{2.5}\right]^{\frac{n}{2-n}} \frac{2^{\frac{n}{2-n}}}{D^{\frac{n}{2-n}}}\left(\frac{2}{\mathbf{Re}_M}\right)^{-\frac{n}{2-n}}\right\}^{s-1} \tag{38}$$

Easy to see that logarithmic derivative:

$$s' = \frac{\partial \ln\left(\frac{\boldsymbol{c}_{fM}}{2}\right)}{\partial \ln\left(\frac{2}{\mathbf{Re}_M}\right)} \tag{39}$$

which is result of differentiation of equation (39), is related with derivative (27) by following equation?

$$s' = s + \frac{n}{2-n}(1-s)\frac{\lambda^n\left[\frac{8K}{\rho}\left(\frac{2n+1}{4n}\right)\left(\frac{2(n+1)}{3n+1}\right)^{2.5}\right]^{\frac{n}{2-n}}}{\left[\lambda^n\left(\frac{8K}{\rho}\left(\frac{2n+1}{4n}\right)\left(\frac{2(n+1)}{3n+1}\right)^{2.5}\right)^{\frac{n}{2-n}} + D^{\frac{n}{2-n}}\right]\left(\frac{1}{\mathbf{Re}_M}\right)^{-\frac{n}{2-n}}} \tag{40}$$

By definition (40) it follows immediately that parameter s' is the exponent of the curve:

$$\frac{\boldsymbol{c}_{fM}}{2} = C'\left(\frac{2}{\mathbf{Re}_M}\right)^{s'} \tag{41}$$

tangential to curve (38) at point corresponding to the selected values of $\mathbf{Re}_M$ and D.

Obtained analogically to (37), substitutive velocity profile of the power-law, drag reducing fluid takes the following form:

$$\frac{v}{v_{max}} = \left[1-\left(\frac{r}{R}\right)\right]^{\frac{1+n}{2n}\frac{1+s'}{s'}} \tag{42}$$

Fig. 1 and Fig. 2 present the influence of exponent index *s*' on the shape and position of curves drawn in coordinate system $[\mathbf{Re}_M, c_{fM}]$ and in the Karman–Prandtl's coordinates system $\left[\mathbf{Re}_M\sqrt{c_{fM}}, 1/\sqrt{c_{fM}}\right]$ Plotted in Fig. 1 and Fig. 2 representative, experimental data [8] obtained for aqueous solution of poly(ethylene oxide) (PEO) flow in pipe with diameter D = 4 mm confirm the validity of the presented theoretical considerations.

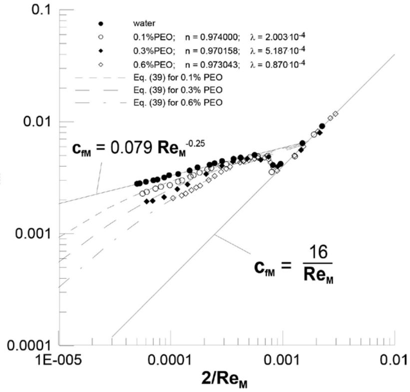

Fig. 1. Comparison of eq. (38) with experimental results

Rys. 1. Konfrontacja równania (38) z wynikami doświadczalnymi

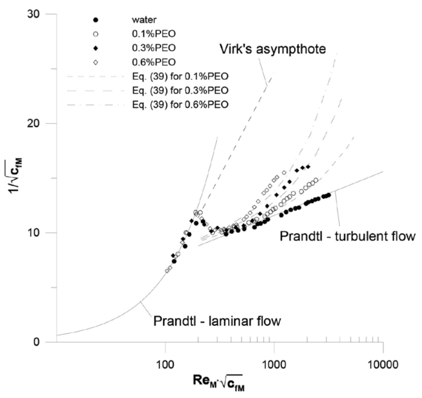

Fig. 2. Comparison of eq. (38) with experimental results presented in Karman coordinates system

Rys. 2. Konfrontacja równania (38) z danymi doświadczalnymi przedstawionymi w układzie współrzędnych Karmana

3. Conclusions

An analysis of the equation (40) and curves presented in Fig. 1 indicates, that for the laminar flow ($s' = s = 1$) velocity profile of the power-law, drag reducing fluid is identical to the velocity profile of power-law, pseudo-rheostable fluid. For the turbulent flow constant s', which characterize the viscoelastic, power-law, Toms fluid, is always greater than the constant s determined for its pseudorheostable, not having elastic properties analogue ($s' > s$). This inequality shows that $(1 + s')/s' < (1 + s)/s$. This means that the power-law, drag reducing fluid velocity profile (42) for turbulent flow is always steeper than velocity profile of rheostable, power-law fluid (37) with the same flow index n.

Symbols

c_f – Fanning friction coefficient [–]
K – flow consistency constant [kg s^{n-2}/m]
L – length of the pipe or capillary [m]
n – flow index [–]
Δp – pressure drop [kg/(s^2m)]
r – radial variable [m]
R – inner radius of the pipe or capillary [m]
Re – Reynolds number [–]
v – local velocity (for turbulent flow – time-averaged local velocity) [m/s]
v_m – mean velocity [m/s]
x – axial variable [m]
λ – characteristic time constant in equation (39) taking into account the viscoelastic properties of fluid [s]
η – Pseudo-Newtonian coefficient of dynamic viscosity [kg/(m s)]
ρ – Fluid density [kg/m^3]
τ – total shear stress [kg/(s^2m)]
' – denotes dimensionless variable
$\bar{}$ – denotes time-averaged quantity
$_M$ – subscript – related with pseudorheostable flow
$_w$ – subscript – concern to the boundary condition ($r=R$)

REMARK: all bolted symbols concern to the pseudo-Newtonian fluid flow

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