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# The curve of pressure in vertically loaded arches

## Linia ciśnień w sklepieniach obciążonych pionowo

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**Słowa kluczowe:** łuk, linia ciśnień

### 1. HISTORICAL CONTRIBUTIONS TO THE CURVE OF PRESSURE

In a plane arch the polygon of pressure is the funicular polygon of the forces, which has as first and last side the restraint reactions. In a generic section across a point P on the geometric axis of the beam (see Fig. 1), the side of the polygon ( $C_{i-1}-C_i$ ) coincides with the vector resultant of the forces situated at right, the opposite of the forces situated at left.

Therefore, if vectors  $e_1$ ,  $e_2$  and  $e_3$  are the intrinsic triad of the arch, the side of the funicular polygon, projected onto  $e_3$  and  $e_2$ , gives respectively the normal stress and the shear, while the moment referred to  $e_1$  is the bending moment.

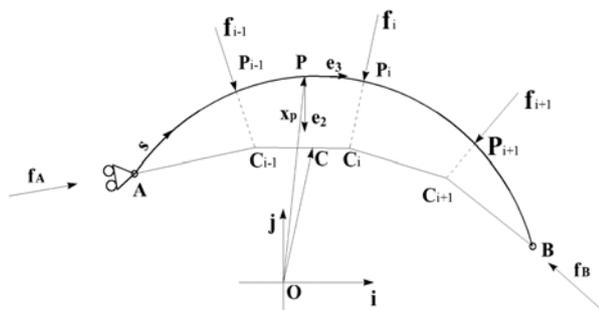


Fig. 1 Polygon of pressure in an arch

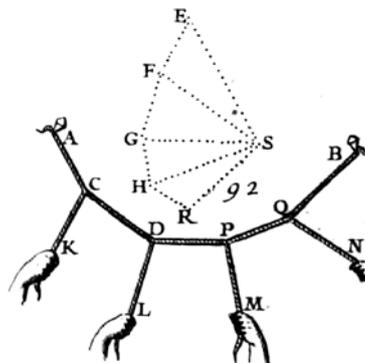


Fig. 2 Varignon's funicular polygon

The origin of the polygon of pressure is related to both M. Varignon [1], who introduced the funicular polygon in 1687 (to find the catenary given the applied forces), Fig. 2, and to the studies of Huygens, Johan Bernoulli and Leibnitz, who independently solved the catenary (the curve of equilibrium of a heavy flexible and inextensible cable), in 1691.

Separately, the English mathematician W. Emerson, 1773, [2], in his Principles of Mechanincs enounced the following proposition LXVI: *If several beams AB,BC,CD, etc. be joined together at B,C,D, and moveable about the points A,B,C,D be placed in a vertical plane, the points A,F, being fixt and through B,C,D, drawing ri, sm, tp perpendicular to the horizon. And if several weights be laid on the angles B,C,D etc. so that the weight on any angle C may be as than all the beams will be kept in equilibrium by these weights.*

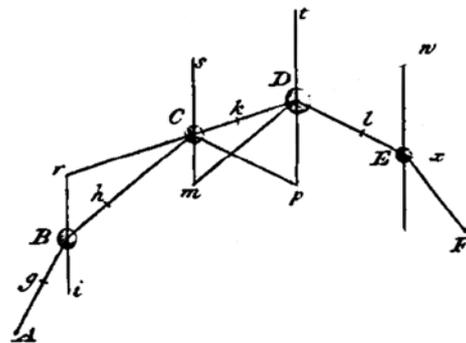


Fig. 3 Emerson's proposition LXVI

Applying Stevin's rule (Fig. 3), the first force  $Bi$  is decomposed along the direction  $g$  and  $h$  in  $BA$  and  $BC$ . For the equilibrium of the beam we have  $Bh = Ch$ . Applying the sinus theorem, after some algebra, Emerson finds the proportion among the equilibrating forces:

$$Bi: \frac{\sin(\hat{A}Bh)}{\sin(\hat{A}Bi)\sin(\hat{i}Bh)} = Cm: \frac{\sin(\hat{h}Ck)}{\sin(\hat{m}Ck)\sin(\hat{h}Cm)} = G \quad (1)$$

$G$  is the value of the constant ratio of (1).

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We note that, curiously, this problem (to find the applied equilibrating forces given the catenary) is the inverse of that of determining the funicular polygon of a plane system of forces already solved by Varignon in 1687 and that no reference is made to Varignon: they differ for an important detail: the first as a degree of freedom, the second three degrees.

The continuous problem of Emerson preposition is solved by Charles Hutton, 1773 [3]. In Section II, "Of the arches", of his essay "Principles of Bridges", in the proposition III Hutton applies Emerson's preposition LXVI to the bridges: his purpose is *To find the proportion of the height (h) of the wall above every part of an arch of equilibration.* He finds (Fig. 4):

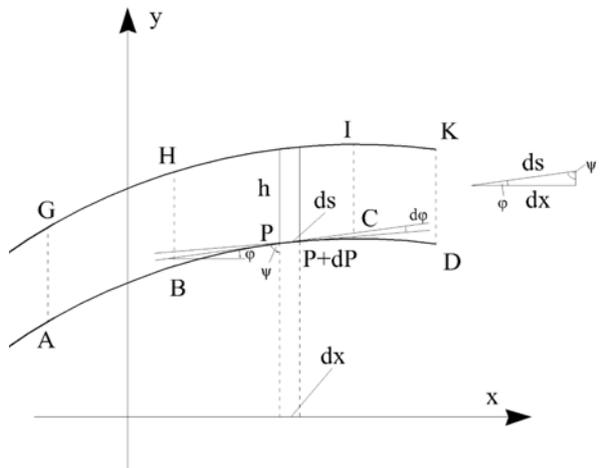


Fig. 4 AD curve of pressure, GK loading arch, according to Hutton

$$h = -\frac{Gd^2 y}{\gamma dx^2} \quad (2)$$

Note that in a Cartesian reference system, vertically oriented,  $y(x)$  is the curve of pressure, that Hutton names arch of equilibration [4],  $\gamma$  is the specific load of the material. Though not explicitly stated by Hutton,  $G$  represents the horizontal thrust in the arch, defined by  $H$  in (7).

The question raised by Hutton proved to be answer to the needs of the arch builders, but his name was rapidly forgotten and outside Great Britain completely unknown. In the continental Europe Coulomb was the leading author for the next hundred years.

Coulomb, 1773, [5] considers two types of collapse mechanisms of a symmetrical voussoir arch. The Fig. 5-Fig. 35, from the original picture from Navier [6], shows the collapse mechanism for the sliding down of the central part of the arch: this fact widens the arch at its bases.

The Fig. 5-Fig. 36 shows the collapse mechanism for the sliding up of the central area of the arch. The Fig. 5-Fig. 37 shows a collapse mechanism for the formation of five (given the symmetry of the arch) hinges respectively on key extrados, on the intrados at about forty-five degrees and on the extrados abutments. The Fig. 5-Fig. 38 shows a collapse mechanism

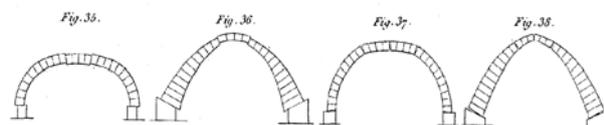


Fig. 5 Mechanism of collapse a stone arch according to Coulomb

for formation of five hinges on key intrados, on the extrados at about forty-five degrees and on the intrados abutments. Coulomb states that the collapse of the arch for sliding is an unlikely hypothesis in practical cases, given the high friction coefficient of the materials used. If  $H$  is the thrust in the key, he determines  $H_{min}$  for the mechanism of Fig. 37 and  $H_{max}$  for the mechanism of Fig. 38. So for the equilibrium of the arch:  $H_{min} < H < H_{max}$ .

Navier, 1823, [7] considers the case of a cable supported in  $A$  (Fig. 6) by two forces, a horizontal one  $H$  and a vertical one  $P$ , submitted to a system of vertical forces function of the place, and the stress  $T$ . A Cartesian orthogonal reference system  $x, y$  has its origin in  $A$ .

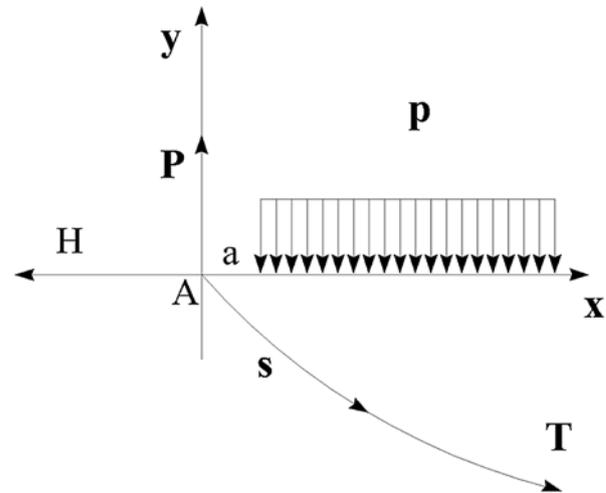


Fig. 6 Equilibrium of a suspension bridge

Navier deals with the suspended bridges, hypothesizing that the forces  $p$  are equally distributed on the horizontal axis, ignoring the curvature of the arch and the weights of guys. Equilibrium yields:

$$T \frac{dx}{ds} = H \quad (3)$$

$$T \frac{dy}{ds} = -P + p(a - x) \quad (4)$$

Substituting (3) in (4) and deriving, he obtains, first:

$$\frac{d^2 y}{dx^2} = -\frac{p}{H} \quad (5)$$

the "funicular curve of a vertical load  $p$ " very similar to (2).

The first author that explicitly defines the curve of pressure (Line of Pressure), is Moseley, 1843, [8]. The historians of the second half of XIX century, like Culmann attribute the paternity of the concept of "curve of pressure" to Moseley (*Der erste, der den Unterschied zwischen Druck- und Stütz-Linie klar und scharf hervorgehoben hat, ist, so viel uns bekannt, Moseley*).

The author of the core of inertia, Culmann 1866, [9] suggests to keep the curve of the pressure inside the core to have an arch fully compressed.

After Culmann a period of more than 100 years of neglect of voussoir arches begins; it was due to the new technologies (steel and reinforced concrete), so that, beginning from Culmann himself (the theory of ellipse of elasticity) and Castigli-

ano [10], research was devoted to linear elastic arch. Especially after the second half of the XX century the neglect was caused by the progressive abandon of arches, in relation to the more economical pre-stressed concrete beams. Heyman [11] has revived the old approach of Coulomb to the voussoir arch, reinterpreting it through a rigid-plastic mechanical model. The hypothesis placed by Heyman, essentially identical to those of Coulomb, are the following:

- The sliding collapse between voussoirs can not occur
- The masonry has no tensile strength
- The masonry has an infinite compressive strength

The third hypothesis is not explicitly contained in Coulomb's memory, but he himself had testified that in specific cases the working stress is much smaller than crashing one. The collapse of the masonry happens by rotation around the extremities of a voussoir, with the formation of a hinge. If  $N$  is the normal stress transmitted from the hinge (which would lead to tensions of infinite value), the moment is (Fig. 7):

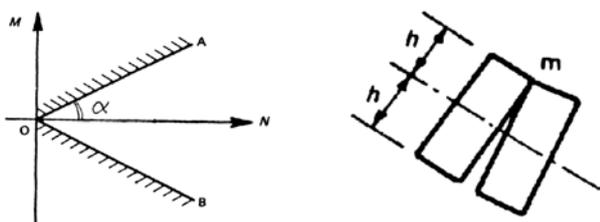


Fig. 7 Heyman's plastic hinge

$$M = \pm Nh \quad (6)$$

The limit domain in the plane  $MN$  is thus represented in Fig. 7.

It follows that it is statically allowable any point within the domain and then, in agreement with the statements of Coulomb, the curve of pressure must be within the masonry. Heyman continues with the statement of Lower Bound Theorem: *if you can find a curve of pressure that lies wholly within the masonry, the arch is safe.*

Unlike the collapse mechanisms applied to ductile structures, which provide a multiplier of loads, and thus introduce a safety factor with respect to load, Heyman suggests a safety geometric factor: *the amount by which the actual arch must shrink to reach its thinnest possible state, such that the curve of pressure is still contained within the arch.*

## 2. THE EQUATION OF THE CURVE OF PRESSURE

We derived, 2010, [4] the vector equation of the funicular curve of a system of forces applied to a plane curve, named "arch", connecting two points called the "supports". The arch, Fig. 7, is loaded by forces represented by plane vectors  $\mathbf{f}_i$ , applied in the points  $P_i$  and the reactions  $\mathbf{f}_A$  and  $\mathbf{f}_B$ .

The restraints, A and B, are such to assure the rigidity of the system and the unique determination of the reactions, represented by two vectors  $\mathbf{f}_A$  and  $\mathbf{f}_B$ . For instance, B may be a fixed hinge and A a shifting hinge.

If we consider the point C of the axis and the system of forces  $\mathbf{f}_B, \mathbf{f}_{1+1}, \dots, \mathbf{f}_1$ , whose resultant is the vector  $(\mathbf{C}_i - \mathbf{C}_{i-1})$  it is clear that:

$$\sum_B^i (\mathbf{P}_k - \mathbf{C}) \times \mathbf{f}_k = \mathbf{0} \quad (7)$$

that is to say, "the resultant moment of the applied forces about any point of the funicular polygon is zero".

In the continuous case the equilibrium of an arch element included between the point P and P+dP is described by the equations:

$$d\mathbf{f} + \mathbf{p} ds = \mathbf{0}, \quad d\mathbf{m} + (\mathbf{P} - \mathbf{O}) \times \mathbf{p} ds = \mathbf{0} \quad (8)$$

From (7) the vector equation defining the funicular, or more properly, the "curve of pressure" is

$$(\mathbf{C} - \mathbf{O}) \times \mathbf{f} = \mathbf{m} \quad (9)$$

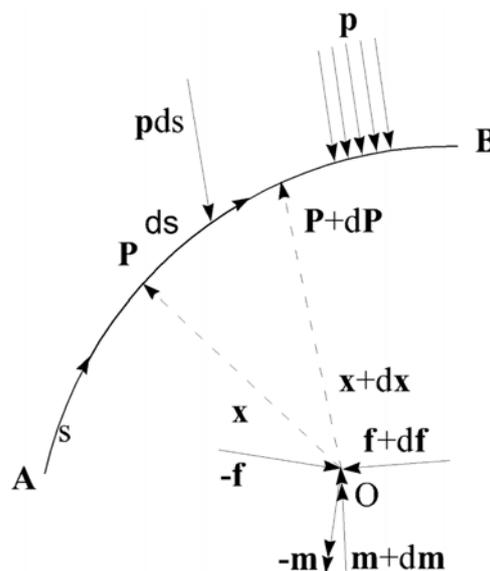


Fig. 8 Equilibrium of an arch

Let us consider the points P and P+dP and the corresponding points on the curve C and C+dx. Differentiating (9) and using (8), we obtain

$$d\mathbf{C} \times \mathbf{f} = \mathbf{0} \quad (10)$$

Then vectors  $d\mathbf{C}$  and  $\mathbf{f}$  are parallel. Developing the vector product we obtain, in the Cartesian reference  $\mathbf{O}, \mathbf{i}, \mathbf{j}$  (Fig. 1), the equation

$$\frac{dy}{dx} = \frac{f_y}{f_x} \quad (11)$$

Differentiation of (11) yields:

$$\frac{d}{ds} \left( \frac{dy}{dx} \right) = \frac{d}{ds} \left( \frac{f_y}{f_x} \right) = \frac{-f_x p_y + f_y p_x}{f_x^2} \quad (12)$$

If  $p_x=0$  and  $p_y=p_y(x)$  we obtain:

$$\frac{d^2 y}{dx^2} + \frac{p_y}{H} = 0 \quad (13)$$

where

$$f_x = H \quad (14)$$

the thrust of the arch, the horizontal constant component of the resultant  $f$  in each point  $P$  of the arch: with the above hypothesis the curve of pressure is independent from the shape of the arch.

This demonstration shows the relationship between the equilibrium equation (8) and (13).

### 3. VERTICALLY LOADED ARCHES

Equation (13) was used mostly in relation to statics of cables, see for instance [12]. Let us show here how structures can be analysed by (13), an equation consequence of (8). The range of integration does not include internal constraints, static or geometric discontinuities, points where boundary conditions must be assigned. If therefore  $n$  is the number of ranges of integration, the unknowns are  $2n+1$ , the number 1 takes into account the static unknown  $H$ , which is unique. Therefore we have determinate and indeterminate structures or structures in which the equilibrium is impossible.

Let us consider Fig. 9, which renders the vertically loaded arch  $\bar{c}$  and its correspondent curve of pressure  $c$ .  $P$  and  $\bar{P}$  are the correspondent points on  $c$  and  $\bar{c}$  respectively. If  $\mathbf{r}$  is the resultant of forces in  $\bar{P}$  and  $H$  is the horizontal constant component of it, the relations are the followings:

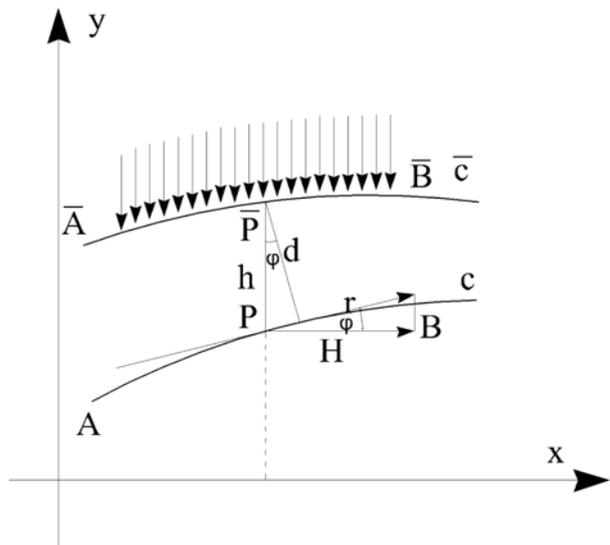


Fig. 9 Curve of pressure  $c$ , and curve of applied vertical loads  $\bar{c}$

$$d = h \cos \varphi = (\bar{y} - y) \cos \varphi \quad (15)$$

$$H = r \cos \varphi \quad (16)$$

$$M = rd = rd \cos \varphi = hH \quad (17)$$

So in vertically loaded arches, the curve of pressure is the diagram of the bending moment  $M$ .

Note also the similarity between Fig. 8 and Fig. 4: this is not surprising because for Hutton the load is proportional to  $h$ , that is natural for the vousoir arch of the XVIII century.

### 3.1. Rigid arch

Let us examine for example the arch of Fig 10. The integration of the equation in two intervals on the right and on the left of the hinge  $C$  leads to four constants which are determined by four conditions, passage for three hinges, so that the curve of pressure is divided into two lines drawn in Fig. 10:

$$y = \pm x + r \quad (18)$$

The fifth unknown  $H$  is determined by the static boundary conditions in the hinge  $C$ , taking into account the inclination of the two lines. So:

$$H = v = \frac{f}{2} \quad (19)$$

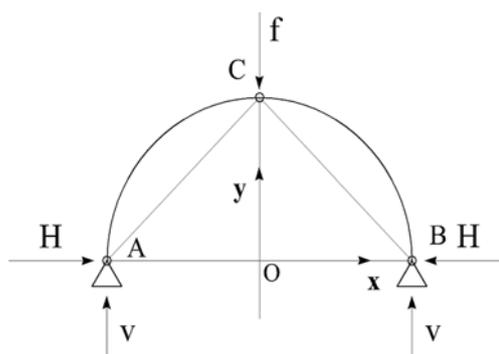


Fig. 10 Three hinges arch

### 3.2. Linear elastic arch

In linear elastic structures the indeterminate constants are determined by congruency conditions. In the following the relations are shown to find rotations  $\Delta\varphi$ , vertical  $\Delta\eta$  and horizontal  $\Delta\xi$  displacements, neglecting the deformation due to normal stress and shear:

$$\Delta\varphi_{AB} = \int_{AB} \frac{M ds}{EI} = H \int_{AB} \frac{h ds}{EI} = H h_0 \int_{AB} \frac{ds}{EI} \quad (20)$$

Note that in (20)  $h_0$  is the centroid of scalars  $\frac{ds}{EI}$ , associated to parallel vectors  $\mathbf{h}$ .

$$\Delta\eta_{AB} = \int_{AB} \frac{M (\bar{x}_B - \bar{x}) ds}{EI} = H \int_{AB} \frac{h (\bar{x}_B - \bar{x}) ds}{EI} \quad (21)$$

$$\Delta\xi_{AB} = \int_{AB} \frac{M (\bar{y}_B - \bar{y}) ds}{EI} = H \int_{AB} \frac{h (\bar{y}_B - \bar{y}) ds}{EI} \quad (22)$$

If  $EI = \text{cost}$ :

$$\int_{AB} \frac{ds}{EI} = \frac{s}{EI} \quad (23)$$

In this case, the relative rotation  $\Delta\varphi_{AB}$  is proportional to the area included by the curve and its curve of pressure.

In the two hinged arch, Fig. 11, the curve of the pressure is divided into two straight lines drawn in the figure, whose equations are:

$$y = \pm \frac{a}{r} x + a \quad (24)$$

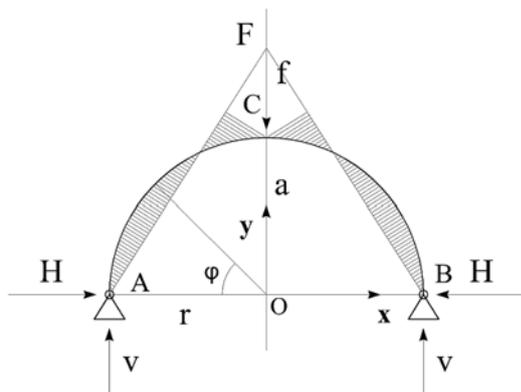


Fig. 11 Two hinges elastic arch

while it is unknown the distance  $a$ . The condition of congruency (no elongation along the diameter AB) can be written as follows, considering (17) the distance of the points of the arch from the curve:

$$2 \int_0^{\frac{\pi}{2}} (-a \cos \varphi - r \sin \varphi + a)(r \sin \varphi) r d\varphi = 0 \quad (25)$$

$$a = r \frac{\pi}{2} \quad (26)$$

and with the boundary conditions in C:

$$v = \frac{f}{2}, \quad H = \frac{f}{\pi} \quad (27)$$

Equation (25) represents the product of inertia of elastic loads with respect to the lines AO and AF.

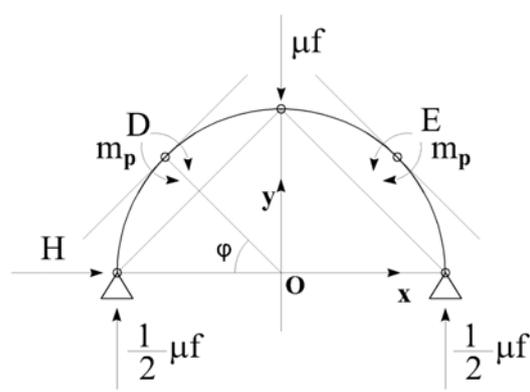


Fig. 12

### 3.3. Rigid-plastic arch

$m_p$  is the plastic moment of a generic section of the arch (Fig. 12). The collapse multiplier  $\mu$  is determined by the equilibrium condition.

$$\mu f = \frac{2m_p}{r(\sin \varphi + \cos \varphi - 1)} \quad (28)$$

and is minimum in the section  $\varphi = \frac{\pi}{4}$  and its symmetric. In these points the curves are discontinuous and have a jump, maintaining their direction.

### 3.4. Coulomb-Heyman arch

With reference to the three hinges arch of Fig. 10 the upper bound of intrados of arch is the line CA and the symmetric line CB.

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## Abstract

This paper is dedicated to the curve of pressure: after a review of its historical contributions, we will show how to analyse structures by its equation.